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ON RECOGNITION BY GRUENBERG–KEGEL GRAPH OF FINITE NONABELIAN SIMPLE GROUPS WITH ORDERS HAVING PRIME DIVISORS AT MOST 13

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The spectrum of a finite group G is the set of all element orders of G. The Gruenberg—Kegel graph (or the prime graph) $\Gamma(G)$ of a finite group G is defined as follows. The vertex set of $\Gamma(G)$ is the set of all prime divisors of the order of G. Two distinct primes p and q are adjacent in $\Gamma(G)$ if and only if there exists an element of order pq in G. We say that the problem of recognition by Gruenberg–Kegel graph (by spectrum, respectively) is solved for a finite group if the number of pairwise non-isomorphic finite groups with the same Gruenberg–Kegel graph (spectrum, respectively) as the group under study is known. In 2005, A. V. Vasil'ev completed solving the problem of recognition by spectrum for all finite nonabelian simple groups with orders having prime divisors at most 13. In this paper we complete the solution of the problem of recognition by Gruenberg–Kegel graph for these groups.

Keywords: finite group, simple group, Gruenberg–Kegel graph (prime graph), recognition

Н. В. Маслова, Л. Г. Нечитайло. О распознавании по графу Грюнберга–Кегеля конечных неабелевых простых групп, простые делители порядков которых не превосходят 13. Спектр конечной группы G — это множество всех порядков элементов группы G. Граф Грюнберга—Кегеля (или граф простых чисел) $\Gamma(G)$ конечной группы G определяется следующим образом. Множество вершин $\Gamma(G)$ — это множество всех простых делителей порядка группы G. Два различных простых числа p и q смежны в $\Gamma(G)$ тогда и только тогда, когда в G существует элемент порядка pq. Мы говорим, что задача распознавания по графу Грюнберга–Кегеля (соответственно, по спектру) решена для конечной группы, если известно число попарно неизоморфных конечных групп с тем же графом Грюнберга–Кегеля (соответственно, спектром), что и у изучаемой группы. В 2005 году А. В. Васильев завершил решение задачи распознавания по спектру для всех конечных неабелевых простых групп с порядками, имеющими простые делители, не превосходящие 13. В данной работе завершается решение задачи распознавания по графу Грюнберга–Кегеля для этих групп.

Ключевые слова: конечная группа, простая группа, граф Грюнберга–Кегеля (граф простых чисел), распознаваемость

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Dedicated to the bright memory of Professor Otto Helmut Kegel

Introduction

Throughout the paper we consider only finite groups and simple graphs, and henceforth the term group means finite group, the term graph means simple graph (undirected graph without loops and multiple edges).

Let G be a group. The *spectrum* $\omega(G)$ of G is the set of all element orders of G. Denote by $\pi(G)$ the set of all prime divisors of $|G|$ (equivalently, the set of all prime elements from $\omega(G)$). The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$. The vertex set of this graph is $\pi(G)$, and two distinct vertices p and q are adjacent in this graph if and only if $pq \in \omega(G)$.

We say that a group G is

- *recognizable* by spectrum (Gruenberg–Kegel graph) if for each group H, $\omega(G) = \omega(H)$ ($\Gamma(G) = \Gamma(H)$, respectively) if and only if $G \cong H$;

- *k-recognizable* by spectrum (Gruenberg–Kegel graph), where k is a positive integer, if there are exactly k pairwise non-isomorphic groups with the same spectrum (Gruenberg–Kegel graph, respectively) as G;
- *almost recognizable* by spectrum (Gruenberg–Kegel graph), if G is k-recognizable by spectrum (Gruenberg–Kegel graph, respectively) for some positive integer k;
- *unrecognizable* by spectrum (Gruenberg–Kegel graph), if there are infinitely many pairwise non-isomorphic groups with the same spectrum (Gruenberg–Kegel graph, respectively) as G.

For a group G we denote by $h_\omega(G)$ and $h_\Gamma(G)$ the number of pairwise non-isomorphic groups H with $\omega(H) = \omega(G)$ and the number of pairwise non-isomorphic groups H with $\Gamma(H) = \Gamma(G)$, respectively. We say that problem of recognition by spectrum (by Gruenberg–Kegel graph) is solved for a group G if the value of $h_\omega(G)$ ($h_\Gamma(G)$, respectively) is known. The problem of recognition by spectrum is well-studied and is solved for the most part of simple groups (see, for example, a survey paper [14]). An overview of the results on recognition by Gruenberg–Kegel graph can be found in [6; 30]. Also in [6], it was proved that a group G is almost recognizable by Gruenberg–Kegel graph if and only if each group H with $\Gamma(H) = \Gamma(G)$ is *almost simple*, i.e. there exists a nonabelian simple group S such that $S \cong Inn(S) \trianglelefteq H \leq \mathrm{Aut}(S)$. In contrast, the problem of recognition by Gruenberg–Kegel graph of almost simple groups is still far from being solved. In [55] A. V. Vasil'ev solved the problem of recognition by spectrum for all nonabelian simple groups G with $\pi(G) \subseteq \{2, 3, 5, 7, 11, 13\}$. We solve the problem of recognition by Gruenberg–Kegel graph for the same set of groups. For many of them the recognition problem has already been solved and the results have already been published, for some of them the results were known, but not published due to their simplicity. We are closing these gaps.

The main result of this paper is the following theorem.

MAIN THEOREM. If G is finite nonabelian simple group such that prime divisors of $|G|$ do not exceed 13, then the problem of recognition by Gruenberg–Kegel graph is solved for G, and the solution is presented in Table 1.

Table 1. Recognition by spectrum and Gruenberg–Kegel graph

G	$\pi(G)$	$h_\omega(G)$	reference	$h_\Gamma(G)$	reference
A_5	$2, 3, 5$	1	[42]	∞	Proposition 3
$L_2(7)$	$2, 3, 7$	1	[40]	∞	Proposition 3
A_6	$2, 3, 5$	∞	[2]	∞	Proposition 1
$L_2(8)$	$2, 3, 5$	1	[3]	∞	Proposition 2
$L_2(11)$	$2, 3, 5, 11$	1	[3]	2	[17, Theorem 3]
$L_2(13)$	$2, 3, 7, 13$	1	[3]	∞	[31, Proposition 5.2]
A_7	$2, 3, 5, 7$	1	[2]	5	Proposition 4
$L_3(3)$	$2, 3, 13$	∞	[34; 45]	∞	Proposition 1
$U_3(3)$	$2, 3, 7$	∞	[33]	∞	Proposition 1
$L_2(25)$	$2, 3, 5, 13$	1	[3]	2	[20, Theorem 3.1]
M_{11}	$2, 3, 5, 11$	1	[48]	2	[17, Theorem 3]
$L_2(27)$	$2, 3, 7, 13$	1	[3]	1	[21, Main Theorem]
A_8	$2, 3, 5, 7$	1	[43]	∞	[51, P. 1008, Remark]
$L_3(4)$	$2, 3, 5, 7$	1	[41]	1	[25, Theorem 9]
$U_4(2)$	$2, 3, 5$	∞	[33]	∞	Proposition 1
$Sz(8)$	$2, 5, 7, 13$	1	[47]	∞	[31, Main Theorem]

Continued on the next page

Table 1: Continuation

G	$\pi(G)$	$h_\omega(G)$	reference	$h_\Gamma(G)$	reference
$L_2(49)$	$2,3,5,7$	1	[3]	5	Proposition 4
$U_3(4)$	$2,3,5,13$	1	[37]	∞	Proposition 5
M_{12}	$2,3,5,11$	1	[48]	∞	[17, Remark 1]
$U_3(5)$	$2,3,5,7$	∞	[33]	∞	Proposition 1
A_9	$2,3,5,7$	1	[39]	∞	Proposition 5
$L_2(64)$	$2,3,5,7,13$	1	[44]	∞	Proposition 2
M_{22}	$2,3,5,7,11$	1	[48]	1	[17, Theorem 3]
J_2	$2,3,5,7$	∞	[36]	∞	Proposition 1
$S_6(2)$	$2,3,5,7$	2	[32; 50]	∞	Proposition 5
A_{10}	$2,3,5,7$	∞	[33]	∞	Proposition 1
$U_4(3)$	$2,3,5,7$	1	[46]	5	Proposition 4
$G_2(3)$	$2,3,7,13$	1	[28]	∞	[31, Main Theorem]
$S_4(5)$	$2,3,5,13$	∞	[34]	∞	Proposition 1
$L_4(3)$	$2,3,5,13$	1	[28]	∞	Proposition 6
$U_5(2)$	$2,3,5,11$	∞	[33]	∞	Proposition 1
${}^2F_4(2)'$	$2,3,5,13$	1	[28]	∞	Proposition 6
A_{11}	$2,3,5,7,11$	1	[39]	∞	[51, Introduction]
$L_3(9)$	$2,3,5,7,13$	2	[38]	∞	Proposition 6
HS	$2,3,5,7,11$	1	[48]	2	[23, Theorem 2]
$S_4(7)$	$2,3,5,7$	∞	[34]	∞	Proposition 1
$O_8^+(2)$	$2,3,5,7$	2	[32; 50]	∞	Proposition 5
${}^3D_4(2)$	$2,3,7,13$	∞	[35]	∞	Proposition 1
A_{12}	$2,3,5,7,11$	1	[32]	∞	Proposition 5
$G_2(4)$	$2,3,5,7,13$	1	[34]	∞	Proposition 5
M^cL	$2,3,5,7,11$	1	[48]	∞	[22, Theorem 5]
$S_4(8)$	$2,3,5,7,13$	∞	[34]	∞	Proposition 1
A_{13}	$2,3,5,7,11,13$	1	[39]	1	[51, Lemma 24]
$S_6(3)$	$2,3,5,7,13$	1	[34]	∞	Proposition 7
$O_7(3)$	$2,3,5,7,13$	2	[50]	∞	Proposition 7
$U_6(2)$	$2,3,5,7,11$	1	[49]	2	[23, Theorem 2]
$U_4(5)$	$2,3,5,7,13$	2	[55]	3	Theorem 1
A_{14}	$2,3,5,7,11,13$	1	[58]	∞	[51, Lemma 26]
$L_5(3)$	$2,3,5,11,13$	1	[8]	2	Theorem 2
Suz	$2,3,5,7,11,13$	1	[48]	1	[23, Theorem 1]
A_{15}	$2,3,5,7,11,13$	1	[58]	∞	[51, Lemma 26]
$O_8^+(3)$	$2,3,5,7,13$	2	[50]	∞	Proposition 7
A_{16}	$2,3,5,7,11,13$	1	[58]	∞	[51, Introduction]
Fi_{22}	$2,3,5,7,11,13$	1	[48]	3	[22, Theorem 3]
$L_6(3)$	$2,3,5,7,11,13$	2	[55]	3	Theorem 3

Crucial steps in the proof of MAIN THEOREM are the following three theorems.

Theorem 1. *The group $U_4(5)$ is 3-recognizable by Gruenberg–Kegel graph.*

Theorem 2. *The group $L_5(3)$ is 2-recognizable by Gruenberg–Kegel graph.*

Theorem 3. *The group $L_6(3)$ is 3-recognizable by Gruenberg–Kegel graph.*

1. Preliminaries

Our terminology and notation are mostly standard and can be found, for example, in [7]. However, it is worth recalling a few definitions for clarity. Let G and H be groups, p be a prime, and π be a set of primes. We denote by $S(G)$ the solvable radical of G (the largest solvable normal subgroup of G), by $F(G)$ the Fitting subgroup of G (the largest nilpotent normal subgroup of G), and by $\Phi(G)$ the Frattini subgroup of G (intersection of all maximal subgroups of G). We denote a semidirect product of G by H as $G : H$ or $G \leftthreetimes H$, by $O_p(G)$ we denote the largest normal p-subgroup of G, by $O_{p'}(G)$ the largest normal subgroup of G whose order is not divisible by p, by $O_\pi(G)$ the largest normal π-subgroup of G, and by $O^p(G)$ the smallest normal subgroup N of G such that G/N is a p-group. Denote the number of connected components of $\Gamma(G)$ by $s(G)$, and the set of connected components of $\Gamma(G)$ by $\{\pi_i(G) \mid 1 \leq i \leq s(G)\}$, for a group G of even order, we assume that $2 \in \pi_1(G)$. Denote by $t(G)$ the independence number of $\Gamma(G)$ (the greatest cardinality of a coclique in $\Gamma(G)$), and by $t(r, G)$ the greatest cardinality of a coclique in $\Gamma(G)$ containing a prime r.

The following assertion is well-known and easy-proving.

Lemma 1. *Let K be a normal subgroup of a group L. Then the following conditions hold*:

(1) *if $r, s \in \pi(K) \setminus \pi(L/K)$ and r and s are non-adjacent in $\Gamma(K)$, then they are also non-adjacent in $\Gamma(L)$*;

(2) *if $r, s \in \pi(L/K) \setminus \pi(K)$, and r and s are non-adjacent in $\Gamma(L/K)$, then they are also non-adjacent in $\Gamma(L)$*;

(3) *if A and B are normal subgroups of a group G such that $A \leq B$ and*

$$r, s \in \pi(B/A) \setminus (\pi(A) \cup \pi(G/B)),$$

then r and s are adjacent in $\Gamma(G)$ if and only if r and s are adjacent in $\Gamma(B/A)$.

The following assertion is also well-known and easy-proving, we provide its proof for completeness.

Lemma 2. *Let G be a finite group such that $G/O_p(G)$ is a simple group for some $p \in \pi(G)$. Let N be a minimal normal subgroup of G such that $N \leq O_p(G)$ with non-trivial action by conjugation of G/N on N. Let $t \in \pi(G) \setminus \{p\}$ and assume that $p \cdot t \notin \omega(G/N)$. Then the following statements hold*:

(1) *N can be considered as a faithfull irreducible $G/O_p(G)$-module*;

(2) *$p \cdot t \in \omega(G)$ if and only if $C_N(\overline{x}) \neq 1$ for some $\overline{x} \in G/O_p(G)$ with $|\overline{x}| = t$.*

Proof. Since $O_p(G)$ is nilpotent, we have $N \leq Z(O_p(G))$. Thus, N can be considered as a $G/O_p(G)$-module. Since N is minimal normal in G, the corresponding action of $G/O_p(G)$ on N is irreducible. Since $G/O_p(G)$ is simple, the corresponding action is faithfull. Thus, statement (1) holds.

If $p \cdot t \in \omega(G)$, then there exists an element $y \in G$ with $|y| = p \cdot t$. Note that $\langle y \rangle \cap N \neq 1$, otherwise $|\langle y \rangle N/N| = |\langle y \rangle|$, therefore $p \cdot t \in \omega(G/N)$; a contradiction. Let $\overline{y}$ be the image of y in $G/O_p(G)$. Then $1 \neq \langle y \rangle \cap N \leq C_N(\overline{y})$.

Let $C_N(\overline{x}) \neq 1$ for some $\overline{x} \in G/O_p(G)$ with $|\overline{x}| = t$. Let x be a preimage of $\overline{x}$ in G. Then $C_N(x) = C_N(\overline{x}) \neq 1$, therefore, $p \cdot t \in \omega(G)$. Thus, statement (2) holds. □

A group G is called a *Frobenius group* if there is a subgroup H of G such that $H \cap H^g = 1$ whenever $g \in G \setminus H$. Let $K = \{1_G\} \cup (G \setminus (\cup_{g \in G} H^g))$ be the *Frobenius kernel* of G.

Lemma 3 (see [1, 35.24 and 35.25], [5] and [52, Theorem 1]). *Let G be a Frobenius group with kernel K and complement H. Then $K \trianglelefteq G$, K is nilpotent, $G = K \rtimes H$, $C_G(h) \leq H$ for each $h \in H \setminus \{1\}$, and $C_G(k) \leq K$ for each $k \in K \setminus \{1\}$. In particular, $K = F(G)$. Moreover, if U is subgroup of order pq in H, where p and q are primes, then U is cyclic. In particular, for any odd prime p, a Sylow p-subgroup of H is cyclic.*

A 2-Frobenius group is a group G which contains a normal Frobenius subgroup R with Frobenius kernel A such that G/A is a Frobenius group with Frobenius kernel R/A.

Lemma 4 [61, Lemma 3]. (1) *If G is a solvable Frobenius group or a 2-Frobenius group, then $\Gamma(G)$ is the union of two connected components each of which is a clique.*

(2) *If G is a non-solvable Frobenius group, then $\Gamma(G)$ is the union of two connected components, one of the which is a complete graph and the other contains the vertices 2, 3, and 5 and is a clique with deleted edge $\{3,5\}$.*

For a group with a disconnected Gruenberg–Kegel graph, the following theorem holds.

Lemma 5 (Gruenberg–Kegel theorem) [57, Theorem A]. *If G is a group with disconnected Gruenberg–Kegel graph, then one of the following statements holds*:

(1) *G is a Frobenius group*;

(2) *G is a 2-Frobenius group*;

(3) *G is an extension of a nilpotent $\pi_1(G)$-group by a group A, where $S \trianglelefteq A \leq \mathrm{Aut}(S)$, S is a nonabelian simple group with $s(G) \leq s(S)$, and A/S is a $\pi_1(G)$-group.*

Lemma 6 [29, Theorem 1]. *Let G be a finite group with $t(G) \geq 3$. Then G is non-solvable.*

We will also need the following result, which generalizes the Gruenberg–Kegel theorem.

Lemma 7 [56]. *Let G be a finite group with $t(G) \geq 3$ and $t(2,G) \geq 2$. Then the following statements hold.*

(1) *There exists a nonabelian simple group S such that $S \trianglelefteq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where K is the solvable radical of G.*

(2) *For every coclique ρ of $\Gamma(G)$ of size at least three, at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(3) *One of the following two conditions holds*:

(3.1) *$S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2,S) = 3$.*

(3.2) *Every prime $p \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$. In particular, $t(2,S) \geq t(2,G)$.*

Lemma 8 [6, Theorem 1.2]. *Let G be a group. The following statements are equivalent*:

(1) *there exist infinitely many groups H such that $\Gamma(G) = \Gamma(H)$;*

(2) *there exists a group H with non-trivial solvable radical such that $\Gamma(G) = \Gamma(H)$.*

Lemma 9 [51, Lemma 1]. *Let N be a normal elementary abelian subgroup of a group G and $H = G/N$. Let $G_1 = N \rtimes H$ be the natural semidirect product of N and H. Then $\Gamma(G) = \Gamma(G_1)$.*

Lemma 10 [10, Lemma 4]. *Let G be a finite simple group, F be a field of characteristic $p > 0$, V be an absolutely irreducible FG-module, and β be a Brauer character of V. If $g \in G$ is an element of prime order distinct from p, then*

$$\dim C_V(g) = (\beta_{\langle g \rangle}, 1_{\langle g \rangle}) = \frac{1}{|g|} \sum_{x \in \langle g \rangle} \beta(x).$$

The following lemma is well-known.

Lemma 11 ([12, Corollary 2.2], see also [9]). *Let $q = 2^n$ and $A \in SL_2(q)$ have trace α and eigenvalues λ, λ^{-1}. Then there exists $P \in SL_2(q)$ such that one of the following statements holds*:

1. $A' = P^{-1}AP = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ *for some* $\mu \in GF(2^n)$ *and the order of* A' *is equal to* 1 *or* 2.

2. $A' = P^{-1}AP = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ *and the order* A' *is a divisor of* $q-1$.

3. $A' = P^{-1}AP = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}$ *and the order of* A' *is a divisor of* $q+1$.

The following lemma is also well-known (see, e. g. [31, Lemma 2.13]).

Lemma 12. *Let* G *be a group,* $g \in G$ *be an element of order* r, *and* ϕ *be a non-trivial irreducible representation of* G *on a non-zero vector space* V. *If the minimum polynomial degree of* $\phi(g)$ *equals to* r, *then* g *fixes in* V *a non-zero vector.*

Lemma 13 (see [32, Lemma 1] and [60, Lemma 2]). *Let* FC *be a Frobenius group with kernel* F *and cyclic complement* $C = \langle c \rangle$. *If* FC *acts faitfully on a vector space* V *over a field of characteristic* p *such that* $(p, |F|) = 1$, *then the minimum polynomial of* c *on* V *is* $x^n - 1$.

Lemma 14 [53, Theorem 1.1]. *Let* G *be one of the groups* ${}^2B_2(q)$, *where* $q > 2$, ${}^2G_2(q)$, *where* $q > 3$, ${}^2F_4(q)$, $G_2(q)$, ${}^3D_4(q)$. *Let* $g \in G$ *be an element of prime power order coprime to* q. *Let* ϕ *be a non-trivial irreducible representation of* G *over a field* F *of characteristic* l *coprime to* q. *Then the minimum polynomial degree of* $\phi(g)$ *equals* $|g|$, *unless possibly when* $G = {}^2F_4(8)$, $l = 3$, $p = 109$ *and* $\phi(1) < 64692$.

A subgroup H is called *pronormal* in a group G if H and H^g are conjugate in $\langle H, H^g \rangle$ for each $g \in G$.

Lemma 15 [15, Lemma 4]. *Let* $H \le A$ *and* $A \trianglelefteq G$. *The following statements are equivalent*:

(1) H *is pronormal in* G;

(2) H *is pronormal in* A *and* $G = AN_G(H)$.

2. Propositions

Proposition 1. *The groups* A_6, $L_3(3)$, $U_3(3)$, $U_4(2)$, $U_3(5)$, J_2, A_{10}, $S_4(5)$, $U_5(2)$, $S_4(7)$, ${}^3D_4(2)$, *and* $S_4(8)$ *are unrecognizable by Gruenberg–Kegel graph.*

Proof. These groups are unrecognizable by spectrum (see Table 1), therefore they are unrecognizable by Gruenberg–Kegel graph. □

Proposition 2. *The group* $L_2(2^n)$ *for* $n \ge 2$ *is unrecognizable by Gruenberg–Kegel graph.*

Proof. Let $q = 2^n$ for $n \ge 2$. Consider the natural action of the group $L = L_2(q) \cong SL_2(q)$ on the natural 2-dimensional vector space V over $GF(q)$. We will show that each element of odd prime order acts on V fixed-point-freely.

Let $A \in SL_2(q)$ be an element of odd prime order r. If A fixes some point of V, then A has an eigenvalue $\lambda = 1$. By Lemma 11, the matrix A is conjugate to one of the following matrices:

- $A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, so $A' = E$, where E is the identity matrix; a contradiction.
- $A' = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}$, so the characteristic polynomial of A is $g(x) = x^2 + \alpha x + 1$. If $\lambda = 1$ is a root of $g(x)$, then $\alpha = 0$, which means $A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In this case, it is easy to see that the order of A' is equal to 2, which leads to a contradiction.

Thus, elements of odd prime orders act on V fixed-point-freely, therefore we get $\Gamma(L) = \Gamma(V \leftthreetimes L)$. From Lemma 8 we obtain that $L = L_2(q)$ is unrecognizable by Gruenberg–Kegel graph. □

In the following propositions we will use results by A. S. Kondrat'ev and I. V. Khramtsov [24;25]. Unfortunately, these papers contain a number of inaccuracies. In our paper we take into account corrections provided in [27] and double check all calculations which require GAP [11].

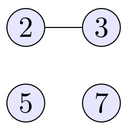
Proposition 3. *Groups A_5 and $L_2(7)$ are unrecognizable by Gruenberg–Kegel graph.*

Proof. Let $G = A_5$. Since $\Gamma(G) = \Gamma(A_6)$ and A_6 is unrecognizable by Gruenberg–Kegel graph, we have that A_5 is also unrecognizable by Gruenberg–Kegel graph.

Let $G = L_2(7)$ or $G = L_2(8)$. By [7], we have $\Gamma(L_2(7)) = \Gamma(L_2(8))$. From Proposition 2 is follows that the group $L_2(8)$ is unrecognizable by Gruenberg–Kegel graph. Thus, the group $L_2(7)$ is also unrecognizable by Gruenberg–Kegel graph. □

Proposition 4. *The groups A_7, $U_4(3)$, and $L_2(49)$ are 5-recognizable by Gruenberg–Kegel graph.*

Proof. By [7], $\Gamma(A_7) = \Gamma(U_4(3)) = \Gamma(L_2(49))$ and is as follows:



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Let G be a group such that $\Gamma(G) = \Gamma(A_7)$. By [25, Theorem 5], one of the following statements holds:

1. G is isomorphic to one of the following groups: M_{11}, $L_2(11)$, $L_3(4) : 2_1$, $L_3(4) : 2_2$, $U_4(3)$, $L_2(25)$, $L_2(25) : 2_1$, $L_2(25) : 2_3$, $S_4(7)$.

2. $G/F(G)$ is isomorphic to $L_2(49)$, $L_2(49) : 2_2$ or $L_2(49) : 2_3$, and $F(G)$ is an abelian 7-group.

3. $G/F(G) \cong A_7$, and $F(G)$ is an elementary abelian 2-group.

We have $11 \in \pi(M_{11})$, $11 \in \pi(L_2(11))$, and $13 \in \pi(L_2(25))$. Also in each of graphs $\Gamma(L_3(4) : 2_2)$, $\Gamma(S_4(7))$ and $\Gamma(L_2(49) : 2_2)$ vertices 2 and 7 are adjacent. By [7], $\Gamma(L_3(4) : 2_1) = \Gamma(A_7)$.

Assume that $G/F(G)$ is isomorphic to $L_2(49)$ or $L_2(49) : 2_3$. We claim that $F(G) = 1$, otherwise $7 \in \pi_1(G)$ by Lemma 5.

Assume that $G/F(G) \cong A_7$. By [26], if vertices 2 and 5 are non-adjacent in $\Gamma(G)$, then $G = O_2(G) \leftthreetimes H$, where $H \cong A_7$, and $O_2(G)$ is the direct product of minimal normal subgroups of G each of which is isomorphic, as a $G/O_2(G)$-module, to one of the two 4-dimensional irreducible $GF(2)A_7$-modules V_i for $i \in \{1, 2\}$ that are conjugate with respect to an outer automorphism of the group A_7. Let P be a Sylow 7-subgroup of G. Then $|P| = 7$ and for each i, P acts on non-zero vectors of the space V_i. We have that $|V_i \setminus \{0\}| = 15$ and lengths of P-orbits are powers of 7. Thus, if vertices 2 and 7 are non-adjacent in $\Gamma(G)$, then $O_2(G) = 1$.

Thus, there are exactly five groups with same Gruenberg–Kegel graph:

$$A_7, L_2(49), L_2(49) : 2_3, U_4(3), L_3(4) : 2_1.$$ □

Proposition 5. *The groups $G_2(4)$, $U_3(4)$, A_9, $S_6(2)$, $O_8^+(2)$, and A_{12} are unrecognizable by Gruenberg–Kegel graph.*

Proof. From the table of 2-modular Brauer characters of the $G_2(4)$ [19] and Lemma 10 we obtain that $\Gamma(G_2(4)) = \Gamma(V \leftthreetimes G_2(4))$, where V is a 6-dimensional absolutely irreducible $GF(4)G_2(4)$-module. Therefore, from Lemma 8 we see that $G_2(4)$ is unrecognizable by Gruenberg–Kegel graph.

From the table of 5-modular Brauer characters of the group $U_3(4)$ [19] and Lemma 10 we obtain that $\Gamma(U_3(4)) = \Gamma(V \leftthreetimes U_3(4))$, where V is a 12-dimensional absolutely irreducible $GF(5)U_3(4)$-module. Therefore, from Lemma 8 we see that $U_3(4)$ is unrecognizable by Gruenberg–Kegel graph.

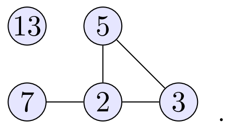
By [7], we have $\Gamma(A_9) = \Gamma(S_6(2)) = \Gamma(O_8^+(2))$. From [61] it follows that there exist solvable groups H_1 and H_2 with $\Gamma(H_1) = \Gamma(A_9)$ and $\Gamma(H_2) = \Gamma(A_{12})$. Thus, by Lemma 8, the groups A_9, $S_6(2)$, $O_8^+(2)$, and A_{12} are unrecognizable by Gruenberg–Kegel graph. $\square$

Proposition 6. *The groups $L_4(3)$, ${}^2F_4(2)'$, $L_3(9)$ are unrecognizable by Gruenberg–Kegel graph.*

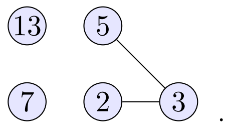
Proof. From [61] it follows that there exist non-solvable Frobenius groups G_1 and G_2 such that $\Gamma(G_1) = \Gamma({}^2F_4(2)') = \Gamma(L_4(3))$ and $\Gamma(G_2) = \Gamma(L_3(9))$. Since by Lemma 3, each Frobenius group has non-trivial solvable radical, by Lemma 8, the groups $L_4(3)$, ${}^2F_4(2)'$, and $L_3(9)$ are unrecognizable by Gruenberg–Kegel graph. $\square$

Proposition 7. [1] *The groups $S_6(3), O_7(3), O_8^+(3)$ are unrecognizable by Gruenberg–Kegel graph.*

Proof. By [7], we have $\Gamma(S_6(3)) = \Gamma(O_7(3)) = \Gamma(O_8^+(3))$, and the graph is as follows:



Consider the group $A = \mathrm{Aut}(Sz(8)) = Sz(8){:}3$. $\Gamma(A)$ is as follows:



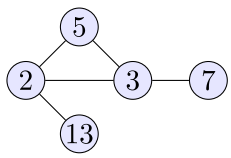
The table of 2-modular Brauer characters of A is available in GAP [11]. Using Lemma 10 one can check that there exists an absolutely irreducible 48-dimensional $GF(2)\mathrm{Aut}(Sz(8))$-module V such that each element of order 13 acts on this module fixed-point-freely, while elements of order 5 and 7 have non-trivial fixed points. Thus,

$$\Gamma(V \rtimes A) = \Gamma(S_6(3)).$$

Then Lemma 8 implies that the groups $S_6(3)$, $O_7(3)$, and $O_8^+(3)$ are unrecognizable by Gruenberg–Kegel graph. $\square$

3. Proof of Theorem 1

Let $G = U_4(5)$. Then by [11], $\Gamma(G)$ is as follows:



Let H be a group such that $\Gamma(H) = \Gamma(G)$. We have $t(2, H) = 2$ and $t(H) = 3$. Then, from Lemma 7, it follows that there exists a nonabelian simple group S such that

$$S \leq \overline{H} = H/K \leq \mathrm{Aut}(S) \text{ for } K = S(H).$$

Moreover, $7 \in \pi(\overline{H}) \setminus \pi(K)$ or $S \cong A_7$ or $L_2(q)$ for some odd q. Clearly, $\pi(S) \subseteq \{2, 3, 5, 7, 13\}$. Using all these restrictions on $\pi(S)$ and [59, Table 1], we get that S is isomorphic to one of the following groups:

$$L_2(q),\ \text{where } q \in \{7, 8, 13, 25, 27, 49, 64\}, U_3(3), U_3(5), L_3(4), J_2, U_4(3), S_4(7), S_6(2), O_8^+(2),$$
$$A_n,\ \text{where } 5 \leq n \leq 10, G_2(3), {}^3D_4(2), Sz(8), U_4(5), L_3(9), S_6(3), O_7(3), G_2(4), S_4(8), O_8^+(3).$$

[1]This proposition was also proved by Nikita Khismatov in his Master Thesis (Ural Federal University, Yekaterinburg, 2022).

We will consider all the possibilities case by case.

Let $S \cong L_2(27)$, $S_4(7)$, ${}^3D_4(2)$, $S_6(3)$, $O_7(3)$, $S_4(8)$, or $O_8^+(3)$. Then in $\Gamma(S)$, vertices 2 and 7 are adjacent, therefore $\Gamma(S)$ is not a subgraph of $\Gamma(H)$, a contradiction.

Let $S \cong L_2(64)$. Then 5 and 13 are adjacent in $\Gamma(S)$, therefore $\Gamma(S)$ it is not a subgraph of $\Gamma(H)$, a contradiction.

Let $S \cong L_3(9)$. Then 7 and 13 are adjacent in $\Gamma(S)$, therefore $\Gamma(S)$ it is not a subgraph of $\Gamma(H)$, a contradiction.

Let $S \cong L_2(7)$, $L_2(8)$, $U_3(3)$, A_5, or A_6. There exists a coclique $\{5, 7, 13\}$ in $\Gamma(H)$. According to Lemma 7, at most one of these numbers can divide the product $|K| \cdot |\overline{G}/S|$. However, we find that $|\{5, 7, 13\} \cap \pi(\mathrm{Aut}(S))| = 1$, this leads to a contradiction.

Let $S \cong A_n$, where $n \in \{7, 8, 9, 10\}$, $L_3(4)$, $L_2(49)$, $U_3(5)$, J_2, $S_6(2)$, $U_4(3)$, $O_8^+(2)$. Since $|\mathrm{Aut}(S)|$ is not divisible by 13, we have $13 \in \pi(K)$. From Lemma 7 we obtain that $5 \notin \pi(K)$ and $7 \notin \pi(K)$. We have $\tilde{K} = K/O_{\{2,3\}}(K) \neq 1$ and let $\tilde{H} = H/O_{\{2,3\}}(K)$. Assume that $\tilde{K} = O_{13}(\tilde{K})$. By [7], S has a subgroup $T \cong L_2(7)$, and T has a subgroup $Y \cong 7:3$ which is a Frobenius group. The kernel of the subgroup Y acts on $O_{13}(\tilde{K})$ fixed-point-freely. From Lemma 13, we have $3 \cdot 13 \in \omega(H)$; a contradiction. Thus, $\tilde{K} \neq O_{13}(\tilde{K})$. An element of order 7 from $\tilde{H}$ acts on $O_{\{2,13\}}(\tilde{K})$ fixed-point-freely, therefore $O_{\{2,13\}}(\tilde{K})$ is nilpotent by the Thompson theorem [52] and $O_{\{2,13\}}(\tilde{K}) = O_{13}(\tilde{K})$. Thus, $O_3(\tilde{K}/O_{13}(\tilde{K})) \neq 1$ and $P = Z(O_3(\tilde{K}/O_{13}(\tilde{K})))$ is a non-trivial normal 3-subgroup of $\tilde{H}/O_{13}(\tilde{K})$. Since 3 and 13 are non-adjacent in $\Gamma(H)$, Lemma 3 implies that P is cyclic. Moreover, each subgroup of P is characteristic in P, therefore, without loss of generality we can assume that $|P| = 3$. Let $C = C_{\tilde{H}/O_{13}(\tilde{K})}(P)$. Since $P \trianglelefteq \tilde{H}/O_{13}(\tilde{K})$, we have $C \trianglelefteq \tilde{H}/O_{13}(\tilde{K})$. Suppose that $5 \in \pi(C)$. We have $5 \notin \pi(K)$ and since $\tilde{H}/\tilde{K} \cong H/K$ is an almost simple group, we obtain that C contains its socle, therefore, $13 \in \pi(C)$ and $13 \cdot 3 \in \omega(H)$, which is not the case. Let P_0 be the preimage of P in $\tilde{K}$. Then P_0 is a normal $\{3, 13\}$-subgroup of $\tilde{H}$ and each element of order 5 from $\tilde{H}$ acts fixed-point-freely on P_0. Thus, P_0 is nilpotent by the Thompson theorem [52] and $3 \cdot 13 \in \omega(H)$; a contradiction.

Let $S \cong L_2(25)$. Since $|\mathrm{Aut}(S)|$ is not divisible by 7, we have $7 \in \pi(K)$. Let $\tilde{H}$ be the smallest factor group of H with the property that 7 divides $|S(\tilde{H})|$. Then $O_7(\tilde{H}) \neq 1$ and 7 does not divide $|\tilde{H}/O_7(\tilde{H})|$. Let T be a Sylow 5-subgroup of $\tilde{H}/O_7(\tilde{H})$ and T_1 be the preimage of T in $\tilde{H}$. By [7], a Sylow 5-subgroup of S is non-cyclic, therefore, T is non-cyclic. By Lemma 3, we have $5 \cdot 7 \in \omega(T_1) \subset \omega(H)$; a contradiction.

Let $S \cong L_2(13)$ or $G_2(3)$. Since by [7], $|\mathrm{Aut}(S)|$ is not divisible by 5, we have 5 in $\pi(K)$. Furthermore, since $3 \cdot 7 \notin \omega(\mathrm{Aut}(S))$, by Lemma 1, at least one of the primes 3 and 7 divides $|K|$. Applying Lemma 7, we obtain that at most one of the primes 5, 7, and 13 divides $|K|$. Since we already have 5 in $\pi(K)$, it follows that 3 is also in $\pi(K)$ and 7 and 13 are not. By the Hall theorems [18, Theorems 3.13 and 3.14], K has a $\{3, 5\}$-Hall subgroup T, and each two $\{3, 5\}$-Hall subgroups of K are conjugate in K. We have T is pronormal in H, therefore, by Frattini's argument (see Lemma 15), $H = KN_H(T)$. Since 13 does not divide $|K|$, there is an element of order 13 in $N_H(T)$ that acts on T fixed-point-freely. By the Thompson theorem [52], T is nilpotent. Replacing an element of order 13 with an element of order 7, we obtain that a $\{2, 5\}$-Hall subgroup of K is also nilpotent.

We now prove that

$$K = N \times R,$$

where R is a Sylow 5-subgroup of K and N is a $\{2, 3\}$-Hall subgroup of K. Since a $\{2, 5\}$-Hall and a $\{3, 5\}$-Hall subgroups of K are nilpotent, we have

$$R \leq C_K(W) \text{ and } R \leq C_K(E),$$

where W and E are a Sylow 2-subgroup and a Sylow 3-subgroup of K, respectively. Thus, $R \leq C_K(\langle W, E\rangle)$, and so $R \leq C_K(N)$. Therefore, we obtain the claimed decomposition.

Let $\tilde{H} = H/(N \times \Phi(R))$ and $\tilde{K} = (N \times \Phi(R))$. By Lemma 9, $\Gamma(\tilde{H}) = \Gamma(\tilde{K} \leftthreetimes \overline{H})$ is a subgraph of $\Gamma(H)$. Without loss of generality, we can assume that $\overline{H} \cong S$ and $\tilde{K}$ is an absolutely irreducible $\overline{H}$-module in characteristic 5. From the table of ordinary characters of S which is available in [7] and Lemma 10, we obtain that each element of order 7 in S has non-trivial centralizer in each absolutely irreducible S-module in characteristic 5. Thus, $7 \cdot 5 \in \omega(\tilde{K} \leftthreetimes \overline{H})$, which contradicts that vertices 7 and 5 are non-adjacent in $\Gamma(H)$.

Let $S \cong Sz(8)$ or $G_2(4)$. Since 2 and 7 are non-adjacent in $\Gamma(G)$, Lemma 7 implies that $7 \notin \pi(K)$. If $5 \in \pi(K)$, then from Lemma 7 we obtain that $13 \notin \pi(K)$. As in the case above, we have

$$K = N \times R,$$

where R is a Sylow 5-subgroup of K and N is a $\{2,3\}$-Hall subgroup of K. Since by Lemma 14 each element of order 7 from S has non-trivial centralizer in each irreducible S-module in characteristic 5, as in the case above, we obtain that $7 \cdot 5 \in \omega(H)$; a contradiction.

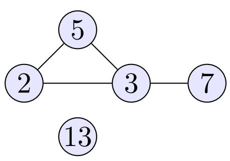
Now suppose that $13 \in \pi(K)$. From Lemma 7 we obtain that $5 \notin \pi(K)$. We have $\tilde{K} = K/O_{\{2,3\}}(K) \neq 1$ and let $\tilde{H} = H/O_{\{2,3\}}(K)$. Note that an element of order 7 from $\tilde{H}$ acts on $O_{\{2,13\}}(\tilde{K})$ fixed-point-freely, and therefore $O_{\{2,13\}}(\tilde{K})$ is nilpotent by the Thompson theorem [52]. In particular, $O_{\{2,13\}}(\tilde{K}) = O_{13}(\tilde{K})$. If $\tilde{K} = O_{13}(\tilde{K})$, then since by Lemma 14 each element of order 7 from S has non-trivial centralizer in each irreducible S-module in characteristic 13, as in the case above, we obtain that $7 \cdot 13 \in \omega(H)$; a contradiction. Thus, $O_3(\tilde{K}/O_{13}(\tilde{K})) \neq 1$ and $P = Z(O_3(\tilde{K}/O_{13}(\tilde{K})))$ is a non-trivial normal 3-subgroup of $\tilde{H}/O_{13}(\tilde{K})$. As above, Lemma 3 implies that P is cyclic and without loss of generality we can assume that $|P| = 3$. Let $C = C_{\tilde{H}/O_{13}(\tilde{K})}(P)$. Since $P \trianglelefteq \tilde{H}/O_{13}(\tilde{K})$, we have $C \trianglelefteq \tilde{H}/O_{13}(\tilde{K})$. Suppose that $5 \in \pi(C)$. We have $5 \notin \pi(K)$ and since $\tilde{H}/\tilde{K} \cong H/K$ is an almost simple group, we obtain that C contains its socle, therefore, $13 \in \pi(C)$ and $13 \cdot 3 \in \omega(H)$, which is not the case. Let P_0 be the preimage of P in $\tilde{K}$. Then P_0 is a normal $\{3, 13\}$-subgroup of $\tilde{H}$ and each element of order 5 from $\tilde{H}$ acts fixed-point-freely on P_0. Thus, P_0 is nilpotent by the Thompson theorem [52]. Since vertices 3 and 13 are non-adjacent in $\Gamma(H)$ and a Sylow 3-subgroup of P_0 is non-trivial, a Sylow 13-subgroup of P_0 is trivial; a contradiction.

Thus, we have K is $\{2,3\}$-group. We claim that $O_2(K)$ is a Sylow 2-subgroup of K. To prove this, we repeat the ideas presented in the proof of [55, Lemma 3.3, Case H]. Let $\tilde{H} = H/O_2(K)$ and $\tilde{K} = K/O_2(K)$. If $\tilde{K} \neq O_3(\tilde{K})$, then $P = Z(O_2(\tilde{K}/O_3(\tilde{K})))$ is a non-trivial normal 2-subgroup of $\tilde{K}/O_3(\tilde{K})$ which acts faithfully on $O_3(\tilde{K})$. As above, let

$$C = C_{\tilde{H}/O_3(\tilde{K})}(P) \trianglelefteq \tilde{H}/O_3(\tilde{K}).$$

Suppose that $13 \in \pi(C)$. As above, in this case we conclude that C has a nonabelian composition factor isomorphic to S, therefore, $7 \in \pi(C)$ and $2 \cdot 7 \in \omega(H)$; a contradiction. If X is a subgroup of order 13 from $\tilde{H}/O_3(\tilde{K})$, then by [13, Theorem 5.2.3], $[P, X]X$ is a Frobenius group with cyclic complement X. By Lemma 13, $13 \cdot 3 \in \omega(H)$; a contradiction. Thus, $O_2(K)$ is a Sylow 2-subgroup of K.

Let $3 \in \pi(K)$. Then $\Gamma(\tilde{H}/\Phi(\tilde{K}))$ is a subgraph of $\Gamma(H)$. By Lemma 9, $\Gamma(\tilde{H}/\Phi(\tilde{K})) = \Gamma(\tilde{K}/\Phi(\tilde{K}) \leftthreetimes \tilde{H}/\tilde{K})$. Lemma 14 implies that $3 \cdot 13 \in \omega(H)$; a contradiction. Thus, K is a 2-group. From [7] we conclude that vertices 3 and 13 are non-adjacent in $\Gamma(\mathrm{Aut}(Sz(8)))$, therefore, by Lemma 1, $S \cong G_2(4)$ and $\Gamma(S)$ is as follows:



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Let H be a group of minimal order with the properties $\Gamma(H) = \Gamma(G)$ and $Soc(H/O_2(H)) \cong G_2(4)$ and let N be a minimal normal subgroup of H with $N \leq K$. Since 2 and 7 are non-adjacent in $\Gamma(H)$, we have that H/N acts non-trivially on N. By Lemma 2, N can be considered as a faithfull irreducible $H/O_2(H)$-module. By the minimality of H we have $H/O_2(H) \cong G_2(4)$ and $C_N(t) \neq 1$ for some $t \in H/O_2(H)$ with $|t| = 13$. Let F be the splitting field of $H/O_2(H)$ over $GF(2)$ and $M = F \otimes_{GF(2)} N$. Then for each $x \in H/O_2(H)$, $C_N(x) \neq 0$ if and only if $C_M(x) \neq 0$ since x has a fixed point if and only if 1 is an eigenvalue of the characteristic polynomial of x. The table of 2-modular Brauer characters of the group $G_2(4)$ is available in [19]. By Lemma 10, for each absolutely irreducible $G_2(4)$-module W in characteristic 2, an element of order 13 has a fixed point in W if and only if an element of order 7 has a fixed point in W, therefore, $2 \cdot 7 \in \omega(H)$; a contradiction. Thus, $K = 1$. Now we can use [7] to verify that $\Gamma(\mathrm{Aut}(G_2(4))) \neq \Gamma(G)$.

Further proof heavily relies on the ideas presented in [55], where the recognizability of the group $U_4(5)$ by spectrum was proved. Now we have $S \cong G$. We aim to prove that $K = 1$. From Lemma 7 we obtain that 7 does not divide $|K|$. Suppose that $13 \in \pi(K)$. Consider the centralizer $C = C_H(K)$. If $C \not\leq K$, then CK/K is a normal subgroup of H/K which is almost simple, therefore, $S \leq CK/K$ and so, 13 is adjacent to each vertex of $\Gamma(H)$; a contradiction. This means that $C \leq K$ and H/K acts faithfully on K. Consider a non-trivial 13-subgroup $P = O_{13}(K/O_{13'}(K))$. Since S has a non-cyclic Sylow 5-subgroup which acts on P, by Lemma 3, we obtain that $13 \cdot 5 \in \omega(H)$; a contradiction. Thus, K is a $\{2,3,5\}$-group.

Since K is solvable, there exists $p \in \{2,3,5\}$ such that $O^p(K) \neq K$. Let

$$\tilde{H} = (H/O^p(K))/\Phi(K/O^p(K)) \text{ and } \tilde{K} = (K/O^p(K))/\Phi(K/O^p(K)).$$

Then $\Gamma(\tilde{H})$ is a subgraph of $\Gamma(H)$, $\tilde{K}$ is an elementary abelian p-group, and by Lemma 9, $\Gamma(\tilde{H}) = \Gamma(\tilde{K} \rtimes \tilde{H}/\tilde{K})$.

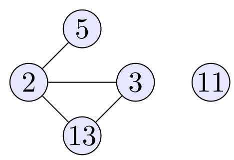
Let $p = 2$. By [4, Table 8.10, 8.33], the group S has a subgroup $T_1 \cong SU_3(5)$. By [54, Theorem B] and Lemma 12, each element of order 7 from T_1 has non-trivial centralizer in each non-trivial absolutely irreducible T_1-module in characteristic 2. Thus, $7 \cdot 2 \in \omega(H)$; a contradiction.

Let $p \in \{3,5\}$. By [4, Table 8.10, 8.33], the group S has a subgroup $T_2 \cong S_4(5)$. From the table of p-modular Brauer characters of the group T_2 [19] and Lemma 10 we obtain that in each absolutely irreducible T_2-module in characteristic p, each element of order 13 has a fixed point (see also [25, Theorem 6]). Thus, $p \cdot 13 \in \omega(H)$; this contradiction completes the proof that $K = 1$.

At the final stage, we have $G \trianglelefteq H \leq \mathrm{Aut}(G)$. Since $|\mathrm{Out}(G)| = 4$, we only need to consider three cases. Using GAP [11], we can check that $\Gamma(G) = \Gamma(U_4(5).2_1) = \Gamma(U_4(5).2_3)$, $\Gamma(G) \neq \Gamma(U_4(5).2_2)$, and $\Gamma(G) \neq \Gamma(\mathrm{Aut}(G))$. Therefore, G is 3-recognizable by Gruenberg–Kegel graph. □

4. Proof of Theorem 2

Let $G = L_5(3)$. Then by [11], $\Gamma(G)$ is as follows:



Let H be a group with $\Gamma(H) = \Gamma(G)$. Since vertices 5, 11, and 13 are pairwise non-adjacent in $\Gamma(H)$, Lemma 6 implies that H is non-solvable. Moreover, by Lemma 4, H is not a non-solvable Frobenius group. Thus, by Lemma 5, $A = H/F(H)$ is an almost simple group. Denote $F(H)$ by K and $Soc(A)$ by S. By Lemma 5, $11 \in \pi(S) \setminus (\pi(K) \cup \pi(A/S))$. Clearly, $\pi(S) \subseteq \pi(G)$. By [59], there are only five simple groups S such that $11 \in \pi(S) \subset \{2,3,5,11,13\}$ which are $U_5(2)$, $L_2(11)$, M_{11}, M_{12}, and $L_5(3)$.

Let $S \cong U_5(2)$. By [7], vertices 5 and 3 are adjacent in $\Gamma(S)$, therefore $\Gamma(S)$ is not a subgraph of $\Gamma(H)$; a contradiction.

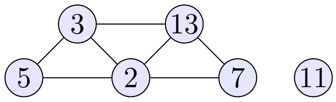
Let $S \cong L_2(11)$, M_{11}, or M_{12}. We have $13 \notin \pi(\mathrm{Aut}(S))$, therefore $13 \in \pi(K)$. Note that by [7], S has a subgroup $X \cong L_2(11)$, and X has a subgroup F isomorphic to a Frobenius group $11 : 5$. Let $\tilde{H} = H/O_{13'}(K)$. Since K is nilpotent, $P = O_{13}(K)$ is non-trivial. Since both 11 and 5 are non-adjacent to 13 in $\Gamma(H)$, F acts on $\tilde{H}$ faithfully, by Lemma 13, we obtain that $13 \cdot 5 \in \omega(H)$; a contradiction.

Thus, $S \cong G$. We aim to prove that $K = 1$. Note that by Lemma 5, $\pi(K) \subseteq \{2, 3, 5, 13\}$. Since K is nilpotent, without loss of generality we can assume K is p-group. By [4, Table 8.18], S has a subgroup F which is a Frobenius group $121 : 5$. If $p = 3$ or 13, as above, F acts on K faithfully, and therefore by Lemma 13, $p \cdot 5 \in \omega(H)$; a contradiction. Suppose Let $p \in \{2, 5\}$. By [54, Theorem B] and Lemma 12, each element of order 11 from F has non-trivial centralizer in each absolutely irreducible F-module in characteristic p. Thus, $11 \cdot p \in \omega(H)$; a contradiction.

Thus, $K = 1$. Now from [11] we obtain that $\Gamma(G) = \Gamma(\mathrm{Aut}(G))$. Thus, G is 2-recognizable by Gruenberg–Kegel graph. □

5. Proof of Theorem 3

Let $G = L_6(3)$. Then by [11], $\Gamma(G)$ is as follows:



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Let H be a group with $\Gamma(H) = \Gamma(G)$. Since vertices 5, 11, and 13 are pairwise non-adjacent in $\Gamma(H)$, by Lemma 6, H is non-solvable. Moreover, by Lemma 4, H is not a non-solvable Frobenius group. Thus, by Lemma 5, $A = H/F(H)$ is an almost simple group. Denote $F(H)$ by K and $Soc(A)$ by S. By Lemma 5, $11 \in \pi(S) \setminus (\pi(K) \cup \pi(A/S))$. Using that $11 \in \pi(S) \subseteq \{2, 3, 5, 7, 11, 13\}$ and [59, Table 1], we find that S is isomorphic to one of the following groups:

$$L_2(11), M_{11}, M_{12}, U_5(2), M_{22}, A_{11}, M^cL, HS, A_{12},$$
$$U_6(2), L_5(3), A_{13}, A_{14}, A_{15}, L_6(3), Suz, A_{16}, Fi_{22}.$$

Let $S \cong A_n$ for $11 \leq n \leq 16$, Suz, or Fi_{22}. By [7], in this case vertices 3 and 7 are adjacent in $\Gamma(S)$, therefore, $\Gamma(S)$ is not a subgraph of $\Gamma(H)$; a contradiction.

Let $S \cong L_5(3)$, M_{11}, M_{12}, or $U_5(2)$. Since 7 does not divide $|\mathrm{Aut}(S)|$, 7 divides $|K|$. From [11] we obtain that a Sylow 3-subgroup of S is non-cyclic. Applying Lemma 3, we deduce that H/K acts on $O_7(K)$ with non-trivial fixed points. Consequently, 3 and 7 are adjacent in $\Gamma(H)$; a contradiction.

Let $S \cong L_2(11)$, HS, $U_6(2)$, M_{22}, or M^cL. Since by [7], $|\mathrm{Aut}(S)|$ is not divisible by 13, we have $13 \in \pi(K)$. Note that by [7] and [4, Table 8.27], S has a subgroup F isomorphic to a Frobenius group $11 : 5$. Let $\tilde{H} = H/O_{13'}(K)$ and $\tilde{K} = K/O_{13'}(K)$. Since K is nilpotent, $P = O_{13}(\tilde{K})$ is non-trivial. Since 11 and 5 is not adjacent to 13 in $\Gamma(H)$, the Frobenius group F acts on P faithfully, and then by Lemma 13 we obtain $13 \cdot 5 \in \omega(G)$; a contradiction.

Thus, we have $S = G$. Now we are going to show that $K = 1$. Without loss of generality we can assume K is a p-group for $p \in \pi_1(H)$. Let $p \neq 3$. Then by [4, Table 8.24], S has a subgroup F which is a Frobenius group $3^5 : 121$ with cyclic complement. Since S is simple we have $C_H(K) \leq K$, then F acts on K faithfully and since $p \neq 3$ we can apply Lemma 13. Thus, $121 \cdot p \in \omega(H)$; a contradiction. Let $p = 3$. Then by [4, Table 8.24], S has a subgroup T isomorphic to $S_6(3)$. By [16, Theorem 1.3], $3 \cdot 7 \in \omega(G)$; a contradiction.

Therefore, we have $G \leq H \leq \mathrm{Aut}(G)$. Since $|\mathrm{Out}(G)| = 4$, we only need to consider four cases. Using GAP [11], we can check that $\Gamma(G) = \Gamma(G\langle\gamma\rangle) = \Gamma(G\langle\delta\gamma\rangle)$ and $\Gamma(G) \neq \Gamma(G\langle\delta\rangle)$, where γ and δ are the standard graph automorphism and diagonal automorphism of G, respectively, and $\Gamma(G) \neq \Gamma(\mathrm{Aut}(G))$. Thus, G is 3-recognizable by Gruenberg–Kegel graph. □

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