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A NOTE ON THE NON-SOLVABLE FORMATION $\hat{\mathfrak{J}_{pr}}$ [1]

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In this paper, we extend the formation $\hat{\mathfrak{J}_{pr}}$, which is generated by the class $\mathfrak{J}_{pr}$ originally introduced by Demina and Maslova. The class $\mathfrak{J}_{pr}$ consists of finite groups in which every non-solvable maximal subgroup has a primary index. Building upon this framework, we introduce and study two generalized formations, denoted by $\hat{\mathfrak{J}}$ and $\hat{\mathfrak{J}_p}$, which are obtained by involving minimal non-solvable maximal subgroups and applying a localization approach to maximal subgroups. We establish new sufficient conditions under which a finite group belongs to these formations. In addition, we give examples of non-solvable groups to illustrate the distinctions between the class $\mathfrak{J}_{pr}$ and its generalizations.

Keywords: formation, non-solvable group, second maximal subgroup, the core of subgroup.

Вэнься Чжоу, Лун Мяо, Байцзюнь Гао, Ран Ли. Замечание о неразрешимой формации $\hat{\mathfrak{J}_{pr}}r$.

В этой статье мы расширяем формацию $\hat{\mathfrak{J}_{pr}}$, которая порождается классом $\mathfrak{J}_{pr}$, первоначально введенным Деминой и Масловой. Класс $\mathfrak{J}_{pr}$ состоит из конечных групп, в которых каждая неразрешимая максимальная подгруппа имеет примарный индекс. Опираясь на эту структуру, мы вводим и изучаем две обобщенные формации, обозначаемые $\hat{\mathfrak{J}}$ и $\hat{\mathfrak{J}_p}$, которые получаются путем включения минимальных неразрешимых максимальных подгрупп и применения локализационного подхода к максимальным подгруппам. Мы устанавливаем новые достаточные условия, при которых конечная группа принадлежит этим формациям. Кроме того, мы приводим примеры неразрешимых групп, иллюстрирующие различия между классом $\mathfrak{J}_{pr}$ и его обобщениями.

Ключевые слова: формация, неразрешимая группа, вторая максимальная подгруппа, ядро подгруппы.

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1. Introduction

Let G be a finite group. Standard notations follow [5; 6]. The notation $A < B$ signifies that A is a proper subgroup of group B, and $A \lessdot B$ denotes that A is a maximal subgroup of B. The core of A in B, denote by A_B, is defined as the largest normal subgroup of B contained in A. Let $Max_2(G)$ denote the set of all second maximal subgroups of G, and $Max(G,H)$ the set of all maximal subgroups of G containing H. A subgroup H is called a strictly second maximal subgroup of G if $H \lessdot M$ for every $M \in Max(G,H)$, and the set of such subgroups is denoted by $Max_2^*(G)$. A subgroup $H \in Max_2(G) \setminus Max_2^*(G)$ is called a weak second maximal subgroup of G.

It is well known that group chains serve as a significant tool for studying group structure. Of particular interest is the concept of maximal chain [2]. Let H denote a subgroup of a group G. A chain of subgroups, $H = M_0 \le M_1 \le M_2 \le \cdots \le M_r = G$, is called a maximal chain joining H to G of length r whenever M_{i-1} is a maximal subgroup of M_i for $i = 1, 2, 3, \ldots, r$. The maximal chain not only reflects the local properties of the group but also provides insights into its global structure. Recall that for a group G and a second maximal subgroup H of G, if $H = 1$, then G is solvable

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(see Lemma 2.4 in [7]). This provides a powerful foundation for studying non-solvable groups by focusing on non-trivial second maximal subgroups. Lytkina and Zhurtov [8] gave a description of finite groups whose maximal subgroups possess only solvable proper subgroups. Denote the class of all such groups by $\mathfrak{X}$. We define the following set $\mathcal{AA}(G)$ and consider the strictly second maximal subgroups within it.

Definition 1. *Let G be a group, we define*

- $\mathcal{AA}(G) = \{H \in Max_2(G) \mid H \textit{ is non-solvable}\}$;
- $\mathcal{AA}^*(G) = \mathcal{AA}(G) \cap Max_2^*(G)$.

Note that if $\mathcal{AA}(G)$ is non-empty, G is non-solvable. This paper is dedicated to the study of non-solvable groups.

By using the idea of classification, we further establish an appropriate collection of maximal subgroups. Recall that if a second maximal subgroup is non-solvable, then all maximal subgroups containing it are also non-solvable. Moreover, Guralnick [4] studied maximal subgroups with prime power index. As an application of the set $\mathcal{AA}(G)$, by combining the non-solvable properties and quantitative characteristics of subgroups, we define the set $\mathcal{B}(G)$ and its corresponding second maximal subgroups $\mathcal{BB}(G)$. Similarly, we consider the strictly second maximal subgroups within it, as follows.

Definition 2. *Let G be a group, we define*

- $\mathcal{B}(G) = \{M \lessdot G \mid M \textit{ is non-solvable}\} \cup \{M \lessdot G \mid |G : M| \textit{ is not a prime power}\}$;
- $\mathcal{BB}(G) = \{H \mid \exists M \in \mathcal{B}(G), s.t., H \lessdot M\}$;
- $\mathcal{BB}^*(G) = \mathcal{BB}(G) \cap Max_2^*(G)$.

Now, we introduce group classes. All maximal subgroups of a solvable group are solvable and have primary indices. However, the converse generally does not hold, with $PSL(2, 7)$ providing an example. Demina and Maslova [3] defined the class $\mathfrak{J}_{pr}$, in which all non-solvable maximal subgroups have prime power index, as follows:

$$\mathfrak{J}_{pr} = \{G \mid \forall M \in Max(G), M \text{ is solvable or } |G : M| \text{ is a prime power}\}.$$

They further explored the possible nonabelian composition factors of non-solvable groups within it. Clearly, $\mathfrak{J}_{pr}$ includes all solvable groups. Building on this, Zhang et al. [13] defined the formation $\hat{\mathfrak{J}_{pr}}$, which is generated by $\mathfrak{J}_{pr}$:

$$\hat{\mathfrak{J}_{pr}} = \langle G \mid G \in \mathfrak{J}_{pr} \rangle.$$

In the light of these work, it is natural to ask if the property of maximal subgroups in $\mathfrak{J}_{pr}$ could be replaced or generalized to other. The answer to this question is positive. Recall that for a group class $\mathcal{F}$, a group G is called minimal non-$\mathcal{F}$-group if $G \notin \mathcal{F}$ while all maximal subgroups of G belongs to $\mathcal{F}$. This general concept specializes to the cases of minimal non-solvable and minimal non-p-solvable groups, among others. Based on this, we define group classes $\mathfrak{J}$ and $\mathfrak{J}_p$ respectively, and obtain corresponding generating formations $\hat{\mathfrak{J}}$ and $\hat{\mathfrak{J}}_p$.

Definition 3. *Let G be a group and p be a prime. We define the following group classes and formations.*

- $\mathfrak{J} = \left\{ G \mid \forall M \in Max(G), \begin{array}{l} M \textit{ is solvable,} \\ \textit{or } M \textit{ is minimal non-solvable} \\ \textit{or } |G : M| \textit{ is a prime power} \end{array} \right\}$;
- $\hat{\mathfrak{J}} = \langle G \mid G \in \mathfrak{J} \rangle$;

- $\mathfrak{J}_p = \left\{ G \mid \forall M \in Max(G), \begin{array}{l} M \text{ is } p\text{-solvable}, \\ \text{or } M \text{ is minimal non-}p\text{-solvable} \\ \text{or } |G:M| \text{ is a prime power} \end{array} \right\};$
- $\hat{\mathfrak{J}}_p = \langle G \mid G \in \mathfrak{J}_p \rangle$.

Clearly, $\mathfrak{J}_{pr} \subseteq \mathfrak{J} \subseteq \mathfrak{J}_p$. Note that $\hat{\mathfrak{J}}$ and $\hat{\mathfrak{J}}_p$ are saturated formations. While the class $\mathfrak{J}$ contains $\mathfrak{J}_{pr}$, the two are different, $\mathfrak{J}$ and $\mathfrak{J}_p$ are also different classes. These two points are demonstrated by the following two examples respectively.

Example 1. *Consider the projective special group $G = PSL(2,9)$ of order 360. The structure and related properties of the maximal subgroups of G are shown in Table 1. A maximal subgroup A_5 has index 6 in G and is non-solvable, but A_5 is minimal non-solvable. Hence, $G \in \mathfrak{J} \backslash \mathfrak{J}_{pr}$.*

Table 1

Maximal subgroups of $PSL(2,9)$

Structure	*Order*	*Solvable*	*Minimal non-solvable*	*Prime power index*
A_5	60	F	T	$F(2 \times 3)$
$3^2:4$	36	T	F	$F(2 \times 5)$
S_4	24	T	F	$F(3 \times 5)$

Example 2. *Consider the Mathieu group $G = M_{11}$ of order 7920. Table 2 shows that $PSL(2,11)$ is not minimal non-solvable but minimal non-11-solvable. Hence, $G \in \mathfrak{J}_p \backslash \mathfrak{J}$.*

Table 2

Maximal subgroups of M_{11}

Structure	*Order*	*Solvable*	*p-Solvable*	*Prime power index*
M_{10}	720	F	T	$T(11)$
$PSL(2,11)$	660	F	F	$F(3 \times 4)$
$M_9:2$	144	F	T	$F(5 \times 11)$
S_5	120	F	T	$F(2 \times 3 \times 11)$
$Q_8:S_3$	48	T	T	$F(3 \times 5 \times 11)$

2. Preliminaries

The following are the main lemmas and definitions we will be concerned with in this paper.

Lemma 2.1 [11, Lemma 2.13]. *Let H be a second maximal subgroup of a group G and $X \in Max(G,H)$. Assume that N is a normal subgroup of G such that $N \leq X$. If $N \nleq H$, then $X = HN$.*

Lemma 2.2. *Let G be a group. If H is a weak second maximal subgroup of G, then there exists a strictly second maximal subgroup of G containing H.*

Proof. Let $S = \{T \in Max_2(G) \mid H \leq T\}$. Let X be an element of S with the largest order. We claim that X is a strictly second maximal subgroup of G. Suppose, for contradiction, that X is a weak second maximal subgroup. There exists $M \in Max(G,X)$ such that X is not maximal in M. Thus, there exists a maximal subgroup Y of M satisfying $H \leq X < Y$. This implies $Y \in S$ and $|Y| > |X|$, which contradicts the choice of X as the element of S with the largest order and the lemma is true. □

Lemma 2.3 [5, Lemma 2.3.4]. *Let N be a normal subgroup of a group G. A subgroup H of a group G is a minimal supplement of N in G if and only if $HN = G$ and $H \cap N \leq \Phi(H)$.*

Lemma 2.4 [1, Theorem 2]. *Let G be a finite group such that, for all primes p, $N_G(P)$ is nilpotent where P is a Sylow p-subgroup of G. Then G is nilpotent.*

Lemma 2.5 [10, Theorem 3.2.2]. *A group G is solvable if and only if $Max_2^*(G) \subseteq T_2(G)$, where $T_2(G) = \{H \mid H \in Max_2(G), \exists M \in Max(G,H), s.t., H_G < M_G\}$.*

Proof. This result was first proved in [10]. We now present a detailed proof for the reader's convenience.

Necessity. Let M be a maximal subgroup of G and H a maximal subgroup of M. Assume that G is solvable and L is a minimal normal subgroup of G. To prove the necessity, it suffices to show that for any strictly second maximal subgroup H satisfying $H_G = M_G$, we have $H \in T_2(G)$.

First suppose the special case $H_G = M_G = 1$. We have $G = LM$ with $L \cap M = 1$ [12, Theorem 5.5 (Baer) Chapter IX]. Clearly, $L \cap H = 1$. Then $HL \lessdot G$. Note that $HL/L \cong H/(H \cap L) \cong H$ and $G/L \cong ML/L \cong M/(M \cap L) \cong M$, by the maximality of H in M, we have $HL/L \lessdot G/L$ and $HL \lessdot G$. Let $M_0 = HL$, we get $H_G < (M_0)_G$.

Now suppose $H_G = M_G \neq 1$. We can assume that $L \leq H_G$ without loss of generality, which implies H/L is a second maximal subgroup of G/L, and G/L is not a group of prime order. Since any finite group of non-prime order has a non-empty set of strictly second maximal subgroups, then $Max_2^*(G/L) \neq \varnothing$. By hypothesis, $Max_2^*(G/L) \subseteq T_2(G/L)$. We claim that $H \in T_2(G)$. If not, then for every $M_1 \in Max(G,H)$, $H_G = (M_1)_G$. Since $L \leq H$, we have $M_1/L \in Max(G/L, H/L)$ and $(H/L)_{G/L} = (M_1/L)_{G/L}$, a contradiction. This completes the proof of necessity.

Sufficiency. We work by induction on $|G|$. Suppose G is a counterexample with minimal order. Let L be a minimal normal subgroup of G. Clearly, $Max_2^*(G) \neq \varnothing$ and G is not simple.

Step I. Verifying that G/L is solvable.

Assume that $Max_2^*(G/L) \neq \varnothing$. Suppose $H/L \in Max_2^*(G/L)$, then $H \in Max_2^*(G)$. By hypothesis, $Max_2^*(G) \subseteq T_2(G)$. Thus, there exists $M_2 \in Max(G,H)$ such that $H_G < (M_2)_G$. Then $M_2/L \in Max(G/L, H/L)$ and $(H/L)_{G/L} < (M_2/L)_{G/L}$. Therefore, $Max_2^*(G/L) \subseteq T_2(G/L)$. Hence, G satisfies the hypothesis of the theorem and G/L is solvable. Thus, L is unique and $\Phi(G) = 1$. If $\Phi(G) \neq 1$, then $L \leq \Phi(G)$. Since $G/\Phi(G) \cong G/L/\Phi(G/L)$. It follows that $G/\Phi(G)$ is solvable and G is solvable, a contradiction.

Step II. Seeking a contradiction.

Let $L_p = L \cap P$, where $P \in Syl_p(G)$. By the Frattini argument, $G = LN_G(L_p)$. But $N_G(L_p) \neq G$, so $G = LN_G(L_p) = LM_3$, where $N_G(L_p) \leq M_3 \lessdot G$. Suppose $H \lessdot M_3$. If $H \in Max_2^*(G)$, then there exists $M_4 \in Max(G,H)$ such that $H_G < (M_4)_G$. By Lemma 2.1, we have $M_4 = HL$. On the other hand, if $H \in Max_2(G) \backslash Max_2^*(G)$, by Lemma 2.2, there exists a second maximal subgroup $I_1 \in Max_2^*(G)$ such that $H < \cdots < I_1 \lessdot M_5 \lessdot G$. There exists $M_6 \in Max(G, I_1)$ such that $H_G \leq (I_1)_G < (M_6)_G$ and then $M_6 = HL$.

In both case, $HL < G$. Then M_3 is a minimal supplement of L in G. By Lemma 2.3, $L \cap M_3 \leq \Phi(M_3)$. Note that, $N_L(L_p) = N_G(L_p) \cap L \leq M_3 \cap L$. It follows that $N_L(L_p) \leq \Phi(M_3)$. Therefore, $N_L(L_p)$ is nilpotent. Applying the generality of p and using Lemma 2.4, we get that L is solvable and G is solvable, a contradiction. □

Definition 4. *Let G be a group and H be a normal subgroup of G. We call H is $\hat{\mathfrak{J}}$-embedded in G if there exists a chief series of G*

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_t (= H) \trianglelefteq \cdots \trianglelefteq H_s = G$$

such that each chief factor H_i/H_{i-1} of G below H belongs to the formation $\hat{\mathfrak{J}}$, where $i = 1, 2, \ldots, s$.

Similarly, we have the definitions of H is $\hat{\mathfrak{J}}_p$-embedded in G and H is $\hat{\mathfrak{J}}_{pr}$-embedded in G.

Lemma 2.6 [9]. *If a finite group G has a nilpotent maximal subgroup M of odd order, then G is solvable.*

Lemma 2.7. *Let L be a minimal normal subgroup of group G. If there exist two proper subgroups of L such that their indices in G are powers of distinct primes, then $L \cong PSL(2,7) \times PSL(2,7) \times \cdots \times PSL(2,7)$.*

Proof. Suppose X_1, X_2 are proper subgroups of L. Let $|L : X_1|$ and $|L : X_2|$ be powers of distinct primes. It follows from $X_1 < L$ and $X_2 < L$ that there exists maximal subgroups L_1, L_2 of L such that $X_1 \leq L_1$ and $X_2 \leq L_2$. Noticing that $|L : L_1| \mid |L : X_1|$ and $|L : L_2| \mid |L : X_2|$, we see the indices of L_1 and L_2 in L are powers of two distinct primes.

Let $L = A_1 \times A_2 \times \cdots \times A_s$, where $A_1 \cong A_2 \cong \cdots \cong A_s$. Without loss of generality, assume $A_1 \nleq L_1$ and $A_2 \nleq L_2$. Clearly, $|L : L_i| = |A_iL_i : L_i| = |A_i : L_i \cap A_i|$ for $i = 1, 2$. Since $L_i \cap A_i < A_i$, there exist maximal subgroups B_i of A_i such that $L_i \cap A_i \leq B_i$ for $i = 1, 2$. Thus, $|A_i : B_i| \mid |A_i : L_i \cap A_i|$ and $|A_i : B_i| \mid |L : L_i|$. In particular, $|A_1 : B_1| \mid |L : L_1|$ when $i = 1$.

Furthermore, there exists an element $g \in G$ such that $A_1 = (A_2)^g$. This implies that $|A_1 : (B_2)^g| = |(A_2)^g : (B_2)^g| = |A_2 : B_2|$, which divides $|L : L_2|$. Therefore, A_1 is a simple group with subgroups of two distinct prime power indices. Then, by a result of Guralnick [4], we see that $A_1 \cong PSL(2,7)$. Hence, $L \cong PSL(2,7) \times PSL(2,7) \times \cdots \times PSL(2,7)$. □

Remark 1. Note that every maximal subgroup of $PSL(2,7)$ is solvable and has primary index. Hence, $PSL(2,7)$ belongs to the class $\mathfrak{J}_{pr}$. Additionally, since $\mathfrak{J}_{pr} \subseteq \hat{\mathfrak{J}}_{pr}$ and $\hat{\mathfrak{J}}_{pr}$ is a saturated formation, it follows that the minimal normal subgroup L in Lemma 2.7 belongs to the formation $\hat{\mathfrak{J}}_{pr}$.

3. Main results

Theorem 3.1. *Let G be a group. If for every second maximal subgroup $H \in \mathcal{AA}^*(G)$, there exists a maximal subgroup $M \in Max(G,H)$ such that $H_G < M_G$, and M_G/H_G is $\hat{\mathfrak{J}}$-embedded in G/H_G, then $G \in \hat{\mathfrak{J}}$.*

Proof. Firstly, suppose that $\mathcal{AA}^*(G) = \varnothing$. For each maximal subgroup M of G, M is solvable or non-solvable. If M is non-solvable, consider any maximal subgroup H of M. If H is non-solvable, then $H \in \mathcal{AA}(G)$, we conclude that $H \in Max_2(G) \backslash Max_2^*(G)$. By Lemma 2.2, there exists a second maximal subgroup $I_1 \in Max_2^*(G)$ such that $H < \cdots < I_1 \lessdot M_0 \lessdot G$. Since H is non-solvable, it follows that I_1 is also non-solvable. In this case, $I_1 \in \mathcal{AA}^*(G)$, a contradiction. Therefore, any maximal subgroup of M is solvable. Consequently, $G \in \mathfrak{X} \subset \mathfrak{J}$.

Next, we can assume that $\mathcal{AA}^*(G) \neq \varnothing$. We work by induction on $|G|$. Suppose G is a counterexample with minimal order. Clearly, G is not simple.

Step I. Prove that if L is a minimal normal subgroup of G, then $G/L \in \hat{\mathfrak{J}}$.

Let L be a minimal normal subgroup of G and consider the quotient group G/L. For any second maximal subgroup $H/L \in \mathcal{AA}^*(G/L)$, H/L is non-solvable and so is H, then $H \in \mathcal{AA}(G)$. Since $H/L \in Max_2^*(G)$, it follows that $H \in Max_2^*(G)$ and $H \in \mathcal{AA}^*(G)$. By hypothesis, there exists a maximal subgroup $M_1 \in Max(G,H)$ such that $H_G < (M_1)_G$ and $(M_1)_G/H_G$ is $\hat{\mathfrak{J}}$-embedded in G/H_G. Then there exists a maximal subgroup $M_1/L \in Max(G/L, H/L)$ such that $(H/L)_{G/L} < (M_1/L)_{G/L}$. Given that $(M_1/L)_{G/L}/(H/L)_{G/L} \cong (M_1)_G/H_G$, it follows that $(M_1/L)_{G/L}/(H/L)_{G/L}$ is $\hat{\mathfrak{J}}$-embedded in $(G/L)/(H/L)_{G/L}$. The minimality of G implies that $G/L \in \hat{\mathfrak{J}}$ and L is unique.

Step II. Verifying that G/L is solvable.

We claim that L is non-solvable. If L is solvable, then $L \in \hat{\mathfrak{J}}$, and since $G/L \in \hat{\mathfrak{J}}$, it follows that $G \in \hat{\mathfrak{J}}$, a contradiction. For any $H/L \in Max_2^*(G/L)$, there exists a maximal subgroup M/L of G/L such that $H/L \lessdot M/L$, implying that $L \leq H \lessdot M \lessdot G$. Then, H is non-solvable and $H \in \mathcal{AA}(G)$. Clearly, $H \in \mathcal{AA}^*(G)$. By hypothesis, there exists a maximal subgroup $M_2 \in Max(G,H)$

satisfying $H_G < (M_2)_G$. For any $H/L \in Max_2^*(G/L)$, there exists a maximal subgroup $M_2/L \in Max(G/L, H/L)$, s.t., $(H/L)_{G/L} < (M_2/L)_{G/L}$. By Lemma 2.5, G/L is solvable.

Step III. Seeking a contradiction.

For any maximal subgroup M of G, there are two possibilities: $L \leq M$ or $L \nleq M$. If $L \leq M$, then $|G : M| = |G/L : M/L|$ is a prime power as G/L is solvable. If $L \nleq M$, we have $G = LM$ and $M_G = 1$. M is either solvable or non-solvable. If M is non-solvable, for any maximal subgroup H of M, we claim that H is solvable. In fact, if H is non-solvable, we get that $H \in \mathcal{AA}(G)$.

We will consider two cases.

1. $H \in Max_2^*(G)$. Then $H \in \mathcal{AA}^*(G)$. By hypothesis, there exists a maximal subgroup $M_3 \in Max(G, H)$ s.t., $H_G < (M_3)_G$ and $(M_3)_G/H_G$ is $\hat{\mathfrak{J}}$-embedded in G/H_G. By Lemma 2.1, $M_3 = HL$. Noticing that $(M_3)_G/H_G \cong (HL)_G/H_G \cong (HL)_G$ and $1 < L \leq (HL)_G$, by Definition 4, we have $L \in \hat{\mathfrak{J}}$ and $G \in \hat{\mathfrak{J}}$, a contradiction.
2. $H \in Max_2(G)\backslash Max_2^*(G)$. By Lemma 2.2, there exists a second maximal subgroup $I_2 \in Max_2^*(G)$ such that $H < \cdots < I_2 \lessdot M_4 \lessdot G$. We have that $H_G = (I_2)_G = 1$. If not, $(I_2)_G \neq 1$, then $(M_4)_G \neq 1$. By Lemma 2.1, $M_4 = HL \leq I_2$, a contradiction. Since I_2 is non-solvable, we have $I_2 \in \mathcal{AA}^*(G)$. By hypothesis, there exists a maximal subgroup $M_5 \in Max(G, I_2)$ s.t., $(I_2)_G < (M_5)_G$ and $(M_5)_G/(I_2)_G$ is $\hat{\mathfrak{J}}$-embedded in $G/(I_2)_G$. Here, $M_5 = I_2L$. Since $(M_5)_G/(I_2)_G \cong (I_2L)_G/(I_2)_G \cong (I_2L)_G$ and $1 < L \leq (I_2L)_G$, we get $L \in \hat{\mathfrak{J}}$ and $G \in \hat{\mathfrak{J}}$, a contradiction.

Hence, H is solvable, as claimed. Therefore, any maximal subgroup of non-solvable group M is solvable. This implies that M is minimal non-solvable.

Consequently, for any maximal subgroup M of G, we obtain that M is solvable, or minimal non-solvable, or $|G : M|$ is a prime power. This implies that $G \in \hat{\mathfrak{J}}$, a contradiction. □

Remark 2. In fact, we are primarily dealing with non-solvable groups. Since $\mathcal{AA}^*(G)$ is the set of non-solvable strictly second maximal subgroups of G, if this set is non-empty, it follows from the definition of solvable groups that G is non-solvable. Therefore, more precisely, in the case $\mathcal{AA}^*(G) \neq \varnothing$ considered in Theorem 3.1, the target groups consist exactly of the non-solvable groups in the formation $\hat{\mathfrak{J}}$.

Apply the idea of localization to the set of second maximal subgroups $\mathcal{AA}(G)$ and the target group class $\mathfrak{J}$, respectively. Specifically, from the perspective of the set $\mathcal{AA}(G)$, we consider the prime factor p of the order of a group G and extract the second maximal subgroups with orders divisible by p from this set. From the viewpoint of target group class, we localize the solvability of the maximal subgroups within $\mathfrak{J}$ to p-solvable, resulting in the group class $\mathfrak{J}_p$.

Next, we seek to establish a connection between the localized second maximal subgroups and the localized group class, leading to the following results.

Theorem 3.2. *Let G be a group. If for every pd-subgroup $H \in \mathcal{AA}^*(G)$, there exists a maximal subgroup $M \in Max(G, H)$ such that $H_G < M_G$, and M_G/H_G is $\hat{\mathfrak{J}}_p$-embedded in G/H_G, then $G \in \hat{\mathfrak{J}}_p$.*

Proof. Assume that the set $\mathcal{AA}^*(G)$ contains no *pd*-subgroup. For any maximal subgroup M of G, M is either solvable or non-solvable. If M is non-solvable, for any maximal subgroup H of M, H is solvable or non-solvable. In the later case, $H \in \mathcal{AA}(G)$. Then we consider the two possibilities. If H is not a *pd*-subgroup, then H is p-solvable. If H is a *pd*-subgroup, by assumption, $H \in Max_2(G)\backslash Max_2^*(G)$. By Lemma 2.2, there exists a second maximal subgroup $I_1 \in Max_2^*(G)$ s.t., $H < \cdots < I_1 \lessdot M_0 \lessdot G$. Noting that I_1 is a *pd*-subgroup and I_1 is non-solvable, contradicts to the hypothesis. Thus, for any non-solvable maximal group M, we conclude that M is either p-solvable or minimal non-p-solvable. Therefore, for any maximal subgroup M of G, we have that either M is p-solvable or minimal non-p-solvable, then $G \in \mathfrak{J}_p$, as desired.

Assume that there exists a pd-subgroup in $\mathcal{AA}^*(G)$. We proceed by induction on $|G|$. Suppose G is a counterexample with minimal order. Clearly, G is not simple. Let L be a minimal normal subgroup of G, it is not difficult to see that $G/L \in \hat{\mathfrak{J}}_p$, and L is unique. Now we prove that G/L is solvable and then seek a contradiction.

Step I. Verifying that G/L is solvable.

Clearly, L is non-solvable. We claim that L is a pd-subgroup. Otherwise, L is p-solvable and $L \in \hat{\mathfrak{J}}_p$. Since $G/L \in \hat{\mathfrak{J}}_p$, it follows that $G \in \hat{\mathfrak{J}}_p$, a contradiction. For any $H/L \in Max_2^*(G/L)$, there exists a maximal subgroup M/L of G/L such that $H/L \lessdot M/L$, implying that $L \leq H \lessdot M \lessdot G$. Then, H is non-solvable and $H \in \mathcal{AA}(G)$. Obviously, $H \in \mathcal{AA}^*(G)$. Since L is a pd-subgroup, we also have H is a pd-subgroup. By hypothesis, there exists a maximal subgroup $M_1 \in Max(G, H)$ satisfying $H_G < (M_1)_G$. Hence, for any $H/L \in Max_2^*(G/L)$, there exists a maximal subgroup $M_1/L \in Max(G/L, H/L)$, s.t., $(H/L)_{G/L} < (M_1/L)_{G/L}$. By Lemma 2.5, we conclude that G/L is solvable.

Step II. Seeking a contradiction.

For any maximal subgroup M of G, there are two possibilities.

1. $L \leq M$. Then $|G : M| = |G/L : M/L|$ is a prime power as G/L is solvable.
2. $L \nleq M$. We have $G = LM$ and $M_G = 1$. M is either p-solvable or non-p-solvable. If M is non-p-solvable, we consider any maximal subgroup H of M. If H is non-p-solvable, then H is a pd-subgroup and $H \in \mathcal{AA}(G)$. Then we consider the following two cases: $H \in Max_2^*(G)$ and $H \in Max_2(G)\backslash Max_2^*(G)$. Similar to the proof of Step III in Theorem 3.1, we see that $L \in \hat{\mathfrak{J}}_p$ and $G \in \hat{\mathfrak{J}}_p$, a contradiction. Hence, for any non-p-solvable subgroup M, every maximal subgroup H of M is p-solvable. Then M is minimal non-p-solvable.

Consequently, for any maximal subgroup M of G, we obtain that M is p-solvable, or minimal non-p-solvable, or $|G : M|$ is a prime power, which implies that $G \in \hat{\mathfrak{J}}_p$, a contradiction. □

As an application of $\mathcal{AA}(G)$, we define the set $\mathcal{BB}(G)$. By analyzing the core relation and index relation among subgroups, we establish a sufficient condition for the formation $\hat{\mathfrak{J}}_{pr}$.

Theorem 3.3. *Let G be a group. If for every second maximal subgroup $H \in \mathcal{BB}^*(G)$, there exists a maximal subgroup $M \in Max(G, H)$ such that $H_G < M_G$, and $|M : H|$ is a prime power, then $G \in \hat{\mathfrak{J}}_{pr}$.*

Proof. We work by induction on $|G|$. Suppose G is a counterexample with minimal order.

(1) Assume that $\mathcal{BB}^*(G) = \varnothing$.

Case I. G is a simple group.

For any maximal subgroup M of G, M is either solvable or non-solvable. If M is non-solvable, for any maximal subgroup H of M, we have $H \in \mathcal{BB}(G)$, then $H \in Max_2(G)\backslash Max_2^*(G)$. By Lemma 2.2, there exists a second maximal subgroup $I_1 \in Max_2^*(G)$ s.t. $H < \cdots < I_1 \lessdot M_1 \lessdot G$. Then $I_1 \notin \mathcal{BB}(G)$, it follows that M_1 is solvable and $|G : M_1|$ is a prime power, set as p^α. There exists a maximal subgroup M_2 such that $P \leq M_2$, where $P \in Syl_p(G)$.

1. $P = M_2$. In this case, $p = 2$. Otherwise, M_2 is a nilpotent maximal subgroup of odd order of G, by Lemma 2.6, G is solvable and $G \in \hat{\mathfrak{J}}_{pr}$, a contradiction. For any $P_1 \lessdot P$, if $P_1 \in Max_2^*(G)$, then $|G : P|$ is a prime power, and let $|G : P| = p_1^{\alpha_1}$. Since $P_1 \lessdot P$ and $|G| = |G : P_1||P_1|$, then $|G|$ is the product of two prime powers and G is solvable, a contradiction. If $P_1 \in Max_2(G)\backslash Max_2^*(G)$, by Lemma 2.2, there exists a second maximal subgroup $I_2 \in Max_2^*(G)$ s.t. $P_1 < \cdots < I_2 \lessdot M_3 \lessdot G$. Similarly, $I_2 \notin \mathcal{BB}(G)$ and $|G : M_3|$ is a prime power. Let $|G|_2 = 2^s$. Given that P_1 is a maximal subgroup of the Sylow 2-subgroup P of G and $P_1 \subset M_3$ with $M_3 \neq P$, we have $|M_3|_2 = 2^{s-1}$. By Lagrange's theorem, $|G| = |G : M_3||M_3|$. It follows that $|G : M_3| = 2$ and $M_3 \trianglelefteq G$, contradicting the case G is a simple group.

2. $P < M_2$. In this case, there exists a maximal subgroup H_1 of M_2 such that $P \leq H_1$. If $H_1 \in Max_2^*(G)$, then $|G : M_2|$ is a prime power, set as $p_2^{\alpha_2}$, with $p_2 \neq p$, then $G \cong PSL(2,7)$ and $G \in \hat{\mathfrak{J}_{pr}}$, a contradiction. If $H_1 \in Max_2(G) \backslash Max_2^*(G)$, there exists a second maximal subgroup $I_3 \in Max_2^*(G)$ s.t. $H_1 < \cdots < I_3 \lessdot M_4 \lessdot G$. Then $|G : M_4|$ is a prime power, which is different from p, then $G \cong PSL(2,7)$ and $G \in \hat{\mathfrak{J}_{pr}}$, a contradiction.

Case II. G is not a simple group.

Let L be a minimal normal subgroup of G and we consider the quotient group G/L. Then the quotient group G/L plainly satisfies the hypothesis of our theorem. By hypothesis, we see that $G/L \in \hat{\mathfrak{J}_{pr}}$ and L is unique. We claim that L is non-solvable. If not, then $L \in \hat{\mathfrak{J}_{pr}}$, and it follows from $G/L \in \hat{\mathfrak{J}_{pr}}$ that $G \in \hat{\mathfrak{J}_{pr}}$, a contradiction.

For any maximal subgroup M of G, M is either solvable or non-solvable. If M is non-solvable, for any maximal subgroup H of M, we have $H \in \mathcal{BB}(G)$, then $H \in Max_2(G) \backslash Max_2^*(G)$. By Lemma 2.2, there exists a second maximal subgroup $I_4 \in Max_2^*(G)$ s.t. $H < \cdots < I_4 \lessdot M_5 \lessdot G$. Then $I_4 \notin \mathcal{BB}(G)$, it follows that M_5 is solvable and $|G : M_5|$ is a prime power, set as q^{β}. We claim that $(M_5)_G = 1$. Otherwise, it follows from the uniqueness of L that $L \leq M_5$. Since L is non-solvable, then M_5 is non-solvable and $I_4 \in \mathcal{BB}^*(G)$, a contradiction. Then $G = LM_5$ and $|G : M_5| = |L : L \cap M_5| = q^{\beta}$.

Using the Frattini argument, we have $G = LN_G(L_q)$ for any $q \in \pi(L)$, here L_q is a Sylow q-subgroup of L. There exists a maximal subgroup M_6 of G such that $N_G(L_q) \leq M_6$ and $(M_6)_G = 1$.

1. If $|G : M_6|$ is a prime power, set as $q_1^{\beta_1}$, then $|G : M_6| = |LM_6 : M_6| = |L : L \cap M_6|$. By Lemma 2.7, $L \cong PSL(2,7) \times PSL(2,7) \times \cdots \times PSL(2,7)$ and $L \in \hat{\mathfrak{J}_{pr}}$, a contradiction.
2. If $|G : M_6|$ is not a prime power, then $M_6 \in \mathcal{B}(G)$. Pick a maximal subgroup H of M_6 containing $(M_6)_q$, where $(M_6)_q \in Syl_q(M_6)$. We have $H \in \mathcal{BB}(G)$, then $H \in Max_2(G) \backslash Max_2^*(G)$ and there exists a second maximal subgroup $I_5 \in Max_2^*(G)$ s.t. $H < \cdots < I_5 \lessdot M_7 \lessdot G$. Similarly, M_7 is solvable and $(M_7)_G = 1$. Thus, $|G : M_7| = |LM_7 : M_7| = |L : L \cap M_7| = q_2^{\beta_2}$, where $q_2 \neq q$. By Lemma 2.7, $L \cong PSL(2,7) \times PSL(2,7) \times \cdots \times PSL(2,7)$ and $L \in \hat{\mathfrak{J}_{pr}}$, a contradiction.

(2) Assume that $\mathcal{BB}^*(G) \neq \varnothing$.

Clearly, G is not simple. We will now establish that the hypothesis is quotient-closed, leading to a contradiction.

Step I. Prove that if L is a minimal normal subgroup of G, then $G/L \in \hat{\mathfrak{J}_{pr}}$.

Let L be a minimal normal subgroup of G, and consider the quotient group G/L. For any second maximal subgroup $H/L \in \mathcal{BB}^*(G/L)$, there exists a maximal subgroup $M/L \in \mathcal{B}(G/L)$ such that $H/L \lessdot M/L$, implying $L \leq H \lessdot M \lessdot G$ and $H \in Max_2^*(G)$. For such an M/L, there are two possibilities:

1. M/L is non-solvable. In this case, M is non-solvable and $H \in \mathcal{BB}^*(G)$.
2. $|G/L : M/L|$ is not a prime power. Since $|G : M| = |G/L : M/L|$, it follows that $M \in \mathcal{B}(G)$ and $H \in \mathcal{BB}^*(G)$.

Therefore, for any second maximal subgroup $H/L \in \mathcal{BB}^*(G/L)$, we get that $H \in \mathcal{BB}^*(G)$. By hypothesis, there exists a maximal subgroup $M_8 \in Max(G,H)$ satisfying $H_G < (M_8)_G$ and $|M_8 : H|$ is a prime power, which also holds for the quotient group G/L. Hence, $G/L \in \hat{\mathfrak{J}_{pr}}$ and L is unique.

Step II. Verifying that G/L is solvable.

Clearly, L is non-solvable. For any $H/L \in Max_2^*(G/L)$, there exists a maximal subgroup M/L of G/L such that $H/L \lessdot M/L$, implying that $L \leq H \lessdot M \lessdot G$. Then, M is non-solvable and $H \in \mathcal{BB}(G)$. Obviously, $H \in \mathcal{BB}^*(G)$. By hypothesis, there exists a maximal subgroup $M_9 \in Max(G,H)$ satisfying $H_G < (M_9)_G$. Then, for any $H/L \in Max_2^*(G/L)$, there exists a maximal subgroup $M_9/L \in Max(G/L, H/L)$, such that $(H/L)_{G/L} < (M_9/L)_{G/L}$. By Lemma 2.5, G/L is solvable.

Step III. If there exists a maximal subgroup L_1 of L, such that the index of L_1 in L is a power of prime, then $L \in \hat{\mathfrak{J}}_{pr}$.

Assume that $|L : L_1| = a^b$. Applying the Frattini argument, for any maximal subgroup M_{10} of G satisfies $L \nleq M_{10}$, we have $G = LN_G(L_a) = LM_{10}$, here L_a is a Sylow a-subgroup of L, $P \leq N_G(L_a) \leq M_{10}$ and $P \in Syl_a(G)$.

We will consider two cases.

1. $|G : M_{10}|$ is a prime power. Let $|G : M_{10}| = a_1^{b_1} (a_1 \neq a)$, then $|G : M_{10}| = |LM_{10} : M_{10}| = |L : L \cap M_{10}| = a_1^{b_1}$. By Lemma 2.7, $L \cong PSL(2,7) \times PSL(2,7) \times \cdots \times PSL(2,7)$ and $L \in \hat{\mathfrak{J}}_{pr}$.
2. $|G : M_{10}|$ is not a prime power. In this case, $M_{10} \in \mathcal{B}(G)$. Pick a maximal subgoup H of M_{10} s.t., $(M_{10})_a \leq H$, where $(M_{10})_a \in Syl_a(M_{10})$, then $H \in \mathcal{BB}(G)$.

   2(a) $H \in Max_2^*(G)$. Then $H \in \mathcal{BB}^*(G)$. By hypothesis, there exists a maximal subgroup $M_{11} \in Max(G, H)$ satisfying $H_G < (M_{11})_G$ and $|M_{11} : H|$ is a prime power. Notice that $L \nleq H$ and $L \leq M_{11}$, by Lemma 2.1, we have $M_{11} = HL$, and $|M_{11} : H| = |HL : H| = |L : L \cap H| = a_2^{b_2}$, with $a_2 \neq a$. By Lemma 2.7, $L \cong PSL(2,7) \times PSL(2,7) \times \cdots \times PSL(2,7)$ and $L \in \hat{\mathfrak{J}}_{pr}$.

   2(b) $H \in Max_2(G) \backslash Max_2^*(G)$. By Lemma 2.2, There exists a second maximal subgroup $I_6 \in Max_2^*(G)$ s.t. $H < \cdots < I_6 \lessdot M_{12} \lessdot G$. If $(M_{12})_G = 1$, then $G = LM_{12}$. If $|G : M_{12}| = |L : L \cap M_{12}|$ is a prime power, then $G \in \hat{\mathfrak{J}}_{pr}$, a contradiction. If $|G : M_{12}|$ is not a prime power, then $I_6 \in \mathcal{BB}^*(G)$. There exists a maximal subgroup $M_{13} \in Max(G, I_6)$ satisfying $(I_6)_G < (M_{13})_G$ and $|M_{13} : I_6|$ is a prime power. Since $(M_{12})_G = 1$, it follows that $(I_6)_G = 1$. By the uniquness of L, we have $L \nleq I_6$. Then Lemma 2.1 yields $M_{13} = I_6 L$. Therefore, we get that $|M_{13} : I_6| = |I_6 L : I_6| = |L : L \cap I_6| = a_3^{b_3}$, where $a_3 \neq a$. Thus, $L \cong PSL(2,7) \times PSL(2,7) \times \cdots \times PSL(2,7)$ and $L \in \hat{\mathfrak{J}}_{pr}$. If $(M_{12})_G \neq 1$, by the uniqueness of L, we have $L \leq M_{12}$. Since L is non-solvable, and so is M_{12}, we have $I_6 \in \mathcal{BB}^*(G)$. By hypothesis, there exists a maximal subgroup $M_{14} \in Max(G, I_6)$ satisfying $(I_6)_G < (M_{14})_G$ and $|M_{14} : I_6| = |L : L \cap I_6|$ is a prime power. Clearly, $L \cong PSL(2,7) \times PSL(2,7) \times \cdots \times PSL(2,7)$ and $L \in \hat{\mathfrak{J}}_{pr}$.

Step IV. Seeking a contradiction.

For any maximal subgroup M of G, there are two possibilities: $L \leq M$ or $L \nleq M$. If $L \leq M$, then $|G : M| = |G/L : M/L|$ is a prime power as G/L is solvable. If $L \nleq M$, we have $G = LM$ and $M_G = 1$. M is either solvable or non-solvable. If M is non-solvable, for any maximal subgroup H of M, we have $H \in \mathcal{BB}(G)$.

We will consider two cases.

1. $H \in Max_2^*(G)$. Then $H \in \mathcal{BB}^*(G)$. By hypothesis, there exists a maximal subgroup $M_{15} \in Max(G, H)$ satisfying $H_G < (M_{15})_G$ and $|M_{15} : H|$ is a prime power. Let $|M_{15} : H| = c_1^{d_1}$. It follows from Lemma 2.1 that $M_{15} = HL$ and $|M_{15} : H| = |HL : H| = |L : L \cap H| = c_1^{d_1}$. By Step III, $L \in \hat{\mathfrak{J}}_{pr}$, leading to $G \in \hat{\mathfrak{J}}_{pr}$, a contradiction.
2. $H \in Max_2(G) \backslash Max_2^*(G)$. By Lemma 2.2, there exists a second maximal subgroup $I_7 \in Max_2^*(G)$ s.t. $H < \cdots < I_7 \lessdot M_{16} \lessdot G$.

Assume that $(M_{16})_G \neq 1$. Since L is unique, $L \leq M_{16}$ and M_{16} is non-solvable. Then, $M_{16} \in \mathcal{B}(G)$ and $I_7 \in \mathcal{BB}^*(G)$. Assume that $(M_{16})_G = 1$. In this case, $(I_7)_G = 1$ and $L \nleq I_7$. If $|G : M_{16}|$ is a prime power, set as $q_2^{\beta_2}$, it follows from $G = LM_{16}$ that $|G : M_{16}| = |LM_{16} : M_{16}| = |L : L \cap M_{16}| = c_2^{d_2}$. By Step III, using Frattini argument, we may get $G \in \hat{\mathfrak{J}}_{pr}$, a contradiction. If $|G : M_{16}|$ is not a prime power, then $I_7 \in \mathcal{BB}^*(G)$.

Thus, $I_7 \in \mathcal{BB}^*(G)$. By hypothesis, there exists a maximal subgroup $M_{17} \in Max(G, I_7)$ satisfying $(I_7)_G < (M_{17})_G$ and $|M_{17} : I_7|$ is a prime power, set as $c_3^{d_3}$. By Lemma 2.1, $M_{17} = I_7L$ and $|M_{17} : I_7| = |I_7L : I_7| = |L : L \cap I_7| = c_3^{d_3}$. Step III implies that $L \in \hat{\mathfrak{J}_{pr}}$ and $G \in \hat{\mathfrak{J}_{pr}}$, a contradiction. □

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