Trudy Instituta Matematiki i Mekhaniki UrO RAN

Vol. 31 No. 4 2025

NONLINEAR ELLIPTIC VARIATIONAL INEQUALITIES WITH UNILATERAL POINTWISE FUNCTIONAL CONSTRAINTS IN VARIABLE DOMAINS[1]

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We consider variational inequalities with operators $\mathcal{A}_s\colon W^{1,p}(\Omega_s) \to (W^{1,p}(\Omega_s))^*$ in divergence form and constraint sets $V_s = \{v \in W^{1,p}(\Omega_s) : h(v) + \Phi_s(v) \leqslant \varphi_s$ a.e. in $\Omega_s\}$, where Ω_s with $s \in \mathbb{N}$ is a domain in $\mathbb{R}^n$ contained in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$, $p > 1$, h is a convex function on $\mathbb{R}$, φ_s is a function on Ω_s, and Φ_s is a continuous convex functional on $W^{1,p}(\Omega_s)$. We describe conditions for a weak convergence of solutions of the considered variational inequalities to the solution of a variational inequality with an operator from $W^{1,p}(\Omega)$ to $(W^{1,p}(\Omega))^*$ and constraint set defined by the equality $V = \{v \in W^{1,p}(\Omega) : h(v)+\Phi(v) \leqslant \varphi$ a.e. in $\Omega\}$, where φ is a limit function for φ_s and Φ is a limit functional for Φ_s. These conditions include some requirements on the involved domains, operators, and the mappings defining the constraint sets. In so doing, one of the main conditions is the G-convergence of the sequence $\{\mathcal{A}_s\}$ to an operator $\mathcal{A}\colon W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$.

Keywords: nonlinear elliptic variational inequality, pointwise functional constraint, variable domains, G-convergence of operators, convergence of solutions.

А. А. Ковалевский. Нелинейные эллиптические вариационные неравенства с односторонними поточечно функциональными ограничениями в переменных областях.

Рассмотрены вариационные неравенства с операторами $\mathcal{A}_s\colon W^{1,p}(\Omega_s) \to (W^{1,p}(\Omega_s))^*$ дивергентного вида и множествами ограничений $V_s = \{v \in W^{1,p}(\Omega_s) : h(v) + \Phi_s(v) \leqslant \varphi_s$ п.в. в $\Omega_s\}$, где Ω_s с $s \in \mathbb{N}$ – область в $\mathbb{R}^n$, содержащаяся в ограниченной области $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$, $p > 1$, h – выпуклая функция на $\mathbb{R}$, φ_s – функция на Ω_s и Φ_s – непрерывный выпуклый функционал на $W^{1,p}(\Omega_s)$. Описаны условия некоторой слабой сходимости решений рассматриваемых вариационных неравенств к решению вариационного неравенства с оператором из $W^{1,p}(\Omega)$ в $(W^{1,p}(\Omega))^*$ и множеством ограничений, определенным равенством $V = \{v \in W^{1,p}(\Omega) : h(v) + \Phi(v) \leqslant \varphi$ п.в. в $\Omega\}$, где φ – предельная функция для φ_s и Φ – предельный функционал для Φ_s. Эти условия включают некоторые требования на участвующие области, операторы и отображения, определяющие множества ограничений. При этом одним из основных условий является G-сходимость последовательности $\{\mathcal{A}_s\}$ к некоторому оператору $\mathcal{A}\colon W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$.

Ключевые слова: нелинейное эллиптическое вариационное неравенство, поточечно функциональное ограничение, переменные области, G-сходимость операторов, сходимость решений.

MSC: 47J20, 49J40, 49J45

DOI: 10.21538/0134-4889-2025-31-4-132-148

1. Introduction

The theory of variational inequalities is an actively developing area of mathematics with numerous applications in various fields (see, e.g., [1; 2] for some results on the solvability of variational inequalities and [2] for their applications). In particular, there is quite an extensive literature devoted to the study of the convergence of solutions of variational inequalities with operators depending on a parameter or having the parameterized domain. This study relates to the theory of multidimensional homogenization, and an important role in it is played by the concept of G-convergence of operators which goes back to the work [3]. The significance of the specified concept is determined first of all by the fact that the G-convergence of a sequence of operators guarantees a certain convergence of solutions of the corresponding operator equations. A detailed investigation of the G-convergence

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[1] This work was partially supported by the Ministry of Science and Higher Education of the Russian Federation within the framework of the Program of development of Ural Federal University under the “Priority-2030” academic leadership program.

and homogenization of linear elliptic and parabolic operators with the same domain was carried out in [4; 5], and the theory of G-convergence and homogenization of nonlinear elliptic and parabolic operators with the same domain was developed in [6; 7]. The G-convergence of nonlinear elliptic operators with variable domain was studied in [8]. It is worth noting the importance of the concept of strong G-convergence of operators in divergence form in the above-mentioned and other related works. In simple words, the G-convergence of a sequence of operators means the weak convergence of the corresponding inverse operators, and the strong G-convergence of operators in divergence form means their G-convergence together with a weak convergence of the so-called generalized gradients defined by the coefficients of the operators.

We emphasize that the G-convergence and the strong G-convergence of operators not only imply a certain convergence of solutions of operator equations and, as a consequence, of solutions of the corresponding boundary value problems, but also these convergences play a key role in establishing the convergence of solutions of variational inequalities with different unilateral and bilateral constraints. Thus, it was shown in [9] that the strong G-convergence of a sequence of linear continuous operators $A_s\colon W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ in divergence form to a similar operator $A\colon W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ implies that the solutions of variational inequalities with the operators A_s and some unilateral and bilateral constraints converge weakly to the solution of the corresponding variational inequality with the operator A and similar constraints. In [10], the weak convergence of solutions of nonlinear variational inequalities with variable measurable unilateral constraints in a fixed domain was established under the assumptions on the strong G-convergence of the corresponding operators and the convergence of the constraint sets in the sense of Mosco. In [8; 11], we proved the weak convergence of solutions of variational inequalities with G-convergent nonlinear elliptic operators and variable regular unilateral and bilateral constraints in variable domains.

We also note that the asymptotic behavior of solutions of variational inequalities with the biharmonic operator and general variable bilateral constraints was studied in the work [12] using Γ-convergence techniques.

We mention some other works on the asymptotic analysis of solutions of variational inequalities with unilateral constraints. Thus, in [13; 14], the convergence of solutions of variational inequalities with nonlinear operators in divergence form (having periodic rapidly oscillating coefficients) and variable regular unilateral constraints in periodically perforated domains and in a fixed domain was studied. In [15; 16], the asymptotic behavior of solutions of variational inequalities for the Laplacian and the p-Laplacian with unilateral constraints on the boundary of perforated domains was investigated, and in [17], the asymptotic behavior of solutions of variational inequalities for the Laplacian and the biharmonic operator with unilateral constraints on subsets located ε-periodically along the domain boundary was studied. Finally, in [18], the asymptotic behavior of solutions of variational inequalities for the p-Laplacian with unilateral constraints on the boundary of perforated domains was investigated in the case of perforations distributed along a manifold.

In the present paper, we also consider variational inequalities with unilateral constraints. However, in contrast to the mentioned and other related works on the asymptotic analysis of solutions of variational inequalities, the constraints we deal with in this work have a more complicated and implicit form.

Speaking in more detail, in the present paper, we consider variational inequalities with operators $\mathcal{A}_s\colon W^{1,p}(\Omega_s) \to (W^{1,p}(\Omega_s))^*$ in divergence form and constraint sets

$$V_s = \{v \in W^{1,p}(\Omega_s) : h(v) + \Phi_s(v) \leqslant \varphi_s \text{ a.e. in } \Omega_s\}, \tag{1.1}$$

where Ω_s with $s \in \mathbb{N}$ is a domain in $\mathbb{R}^n$ contained in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$, $p > 1$, h is a convex function on $\mathbb{R}$, φ_s is a function on Ω_s, and Φ_s is a continuous convex functional on $W^{1,p}(\Omega_s)$. We describe conditions for a weak convergence of solutions of the considered variational inequalities to the solution of a variational inequality with an operator from $W^{1,p}(\Omega)$ to $(W^{1,p}(\Omega))^*$ and constraint set defined by the equality

$$V = \{v \in W^{1,p}(\Omega) : h(v) + \Phi(v) \leqslant \varphi \text{ a.e. in } \Omega\},$$

where φ is a limit function for φ_s and Φ is a limit functional for Φ_s. These conditions include some requirements on the involved domains, operators, and the mappings defining the constraint sets. In so doing, one of the main conditions is the G-convergence of the sequence $\{\mathcal{A}_s\}$ to an operator $\mathcal{A}\colon W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$.

We note that the convergence of solutions of minimum problems for integral and more general functionals $I_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ on sets defined by unilateral pointwise functional constraints including sets of the form (1.1) was studied in our recent works [19; 20].

The structure of the present paper is as follows. In Section 2, we state necessary assumptions and definitions. In Section 3, we introduce the operators $\mathcal{A}_s$ involved in the variational inequalities considered in the sequel. We describe some properties of these operators and prove a result on the existence of some special sequences of functions approximating in $W^{1,p}$-norms the solutions of equations with the operators $\mathcal{A}_s$ and the right-hand sides related to the values of the G-limit operator of the sequence $\{\mathcal{A}_s\}$ (see Proposition 1). In Section 4, we introduce the constraint sets V_s and V involved in the studied variational inequalities and in the corresponding limit variational inequality, respectively. This is preceded by the description of the conditions on the functions and functionals defining the sets V_s and V. After introducing the specified sets, we establish Propositions 2 and 3 which play a key role in the proof of our main result (Theorem 1) stated in Section 5. We note that while Proposition 2 concerns a relation only between the sets V_s and V, Proposition 3 describes a relation between V_s and V involving the operators $\mathcal{A}_s$ and their G-limit. We also note that Proposition 1 is essentially used in the proof of Proposition 3. As for Theorem 1, it not only asserts that solutions u_s of variational inequalities with the operators $\mathcal{A}_s$ and the sets V_s converge weakly to the solution of the limit variational inequality but also describes the convergence of the functionals $\mathcal{A}_s u_s$ and the energy integrals $\langle \mathcal{A}_s u_s, u_s\rangle$. In addition, this theorem asserts that the solutions u_s are approximated in $W^{1,p}$-norms by the solutions of equations with the operators $\mathcal{A}_s$ and the right-hand sides related to the value of the G-limit operator on the solution of the limit variational inequality. The proof of Theorem 1 is based on the use of the approach to the study of the convergence of solutions of variational inequalities with abstract operators proposed in [8, Subsection 1.4]. For some applications of this approach to variational inequalities with nonlinear elliptic operators, see, e.g., [8, Subsection 2.5] and [11, Section 3]. Finally, in Section 6, we give some examples where the conditions on the functions and the functionals defining the constraint sets V_s and V are satisfied.

2. Assumptions and definitions

Let $n \in \mathbb{N}$, $n \geqslant 2$, let Ω be a bounded domain in $\mathbb{R}^n$, and let $p > 1$. Let $\{\Omega_s\}$ be a sequence of domains in $\mathbb{R}^n$ contained in Ω.

We recall some definitions used in the study of the limit behavior of solutions of minimization problems, boundary value problems, and variational inequalities in the domains Ω_s (see, e.g., [11]).

Definition 1. We say that the sequence of domains Ω_s exhausts the domain Ω if for every increasing sequence $\{m_j\} \subset \mathbb{N}$, we have

$$\operatorname{meas}\Big(\Omega \setminus \bigcup_{j=1}^{\infty} \Omega_{m_j}\Big) = 0.$$

It is easy to see that if $v \in W^{1,p}(\Omega)$ and $s \in \mathbb{N}$, then $v|_{\Omega_s} \in W^{1,p}(\Omega_s)$.

For every $s \in \mathbb{N}$, let $q_s\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega_s)$ be the mapping such that for every function $v \in W^{1,p}(\Omega)$, $q_s v = v|_{\Omega_s}$.

Definition 2. We say that the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$ if there exists a sequence of linear continuous operators $l_s\colon W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and for every $s \in \mathbb{N}$ and every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in Ω_s.

We denote by $\mathcal{H}$ the set of all sequences $\{v_s\}$ such that for every $s \in \mathbb{N}$, $v_s \in W^{1,p}(\Omega_s)$.

Definition 3. Let $\{v_s\} \in \mathcal{H}$. We say that the sequence $\{v_s\}$ is bounded if the sequence of norms $\|v_s\|_{W^{1,p}(\Omega_s)}$ is bounded.

Definition 4. Let $\{v_s\} \in \mathcal{H}$, and let $v \in W^{1,p}(\Omega)$. We say that the sequence $\{v_s\}$ converges weakly to the function v if the sequence $\{v_s\}$ is bounded and $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$.

It is easy to see that if $v \in W^{1,p}(\Omega)$, then $\{q_s v\} \in \mathcal{H}$ and the sequence $\{q_s v\}$ converges weakly to the function v.

We denote by $\mathcal{H}^*$ the set of all sequences $\{f_s\}$ such that for every $s \in \mathbb{N}$, $f_s \in (W^{1,p}(\Omega_s))^*$.

Definition 5. Let $\{f_s\} \in \mathcal{H}^*$, and let $f \in (W^{1,p}(\Omega))^*$. We say that the sequence $\{f_s\}$ converges strongly to the functional f if for every function $v \in W^{1,p}(\Omega)$ and every sequence $\{v_s\} \in \mathcal{H}$ converging weakly to the function v, we have $\langle f_s, v_s\rangle \to \langle f, v\rangle$.

We note that Definitions 3–5 are realizations of the corresponding definitions given in [8, Section 1] for the case of abstract Banach spaces.

Further, we assume that the following conditions are satisfied:

(C_1) the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact;

(C_2) the sequence of domains Ω_s exhausts the domain Ω;

(C_3) the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$.

We denote by $\mathcal{L}$ the set of all sequences $\{l_s\}$ such that:

(a) for every $s \in \mathbb{N}$, l_s is a linear continuous operator from $W^{1,p}(\Omega_s)$ to $W^{1,p}(\Omega)$;

(b) the sequence of norms $\|l_s\|$ is bounded;

(c) for every $s \in \mathbb{N}$ and every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in Ω_s.

As seen, condition (C_3) implies that the set $\mathcal{L}$ is nonempty. We also note that if $f \in (W^{1,p}(\Omega))^*$ and $\{l_s\} \in \mathcal{L}$, then $\{f \circ l_s\} \in \mathcal{H}^*$. On the whole, conditions (C_1)–(C_3) make it possible to obtain a number of important assertions concerning the sequences in the sets $\mathcal{H}$ and $\mathcal{H}^*$. These assertions are used in the sequel, and for a detailed exposition of them, see, e.g., [11, Section 2]. In particular, according to Proposition 2.9 in [11], if $f \in (W^{1,p}(\Omega))^*$ and $\{l_s\} \in \mathcal{L}$, then the sequence $\{f \circ l_s\}$ converges strongly to the functional f.

Definition 6. For every $s \in \mathbb{N}$, let $A_s\colon W^{1,p}(\Omega_s) \to (W^{1,p}(\Omega_s))^*$ be an invertible operator, and let $A\colon W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be an invertible operator. We say that the sequence $\{A_s\}$ G-converges to the operator A if for every functional $f \in (W^{1,p}(\Omega))^*$ and every sequence $\{f_s\} \in \mathcal{H}^*$ converging strongly to the functional f, the sequence $\{A_s^{-1} f_s\}$ converges weakly to the function $A^{-1} f$.

This definition is a particular case of the general definition of G-convergence of abstract operators with variable domain given in [8, Section 1].

3. Operators $\mathcal{A}_s$ and their properties

In this section, we introduce the operators involved in the variational inequalities studied in the present paper. We begin with the conditions on the coefficients of these operators.

Let $0 < p_1 \leqslant \min\{p, p/(p-1)\}$, let $p_2 \geqslant \max\{p, 2\}$, and let $c_1, c_2 > 0$. For every $s \in \mathbb{N}$ and every $i \in \{1, \dots, n\}$, let $a_i^{(s)}\colon \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. We assume that for every $s \in \mathbb{N}$, every $x \in \Omega_s$, and every $\xi, \xi' \in \mathbb{R}^n$, the following relations hold:

$$\sum_{i=1}^{n} |a_i^{(s)}(x, 0)| = 0, \tag{3.1}$$

$$\sum_{i=1}^{n} |a_i^{(s)}(x, \xi) - a_i^{(s)}(x, \xi')|^{p/(p-1)} \leqslant c_1 (1 + |\xi| + |\xi'|)^{p-p_1} |\xi - \xi'|^{p_1}, \tag{3.2}$$

$$\sum_{i=1}^{n}[a_i^{(s)}(x,\xi)-a_i^{(s)}(x,\xi')](\xi_i-\xi_i')\geqslant c_2(1+|\xi|+|\xi'|)^{p-p_2}|\xi-\xi'|^{p_2}. \tag{3.3}$$

Next, let $c_3, c_4 > 0$, and, for every $s\in\mathbb{N}$, let $a_0^{(s)}\colon \Omega_s\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function. We assume that for every $s\in\mathbb{N}$, every $x\in\Omega_s$, and every $\eta,\eta'\in\mathbb{R}$, the following relations hold:

$$a_0^{(s)}(x,0)=0, \tag{3.4}$$

$$|a_0^{(s)}(x,\eta)-a_0^{(s)}(x,\eta')|^{p/(p-1)}\leqslant c_3(1+|\eta|+|\eta'|)^{p-p_1}|\eta-\eta'|^{p_1}, \tag{3.5}$$

$$[a_0^{(s)}(x,\eta)-a_0^{(s)}(x,\eta')](\eta-\eta')\geqslant c_4(1+|\eta|+|\eta'|)^{p-p_2}|\eta-\eta'|^{p_2}. \tag{3.6}$$

It is easy to see that relations (3.1)–(3.3) imply that for every $s\in\mathbb{N}$, every $x\in\Omega_s$, and every $\xi\in\mathbb{R}^n$,

$$\sum_{i=1}^{n}|a_i^{(s)}(x,\xi)|^{p/(p-1)}\leqslant c_1(1+|\xi|)^p, \tag{3.7}$$

$$\sum_{i=1}^{n}a_i^{(s)}(x,\xi)\xi_i\geqslant 2^{p-p_2}c_2(|\xi|^p-1). \tag{3.8}$$

Similarly, relations (3.4)–(3.6) imply that for every $s\in\mathbb{N}$, every $x\in\Omega_s$, and every $\eta\in\mathbb{R}$,

$$|a_0^{(s)}(x,\eta)|^{p/(p-1)}\leqslant c_3(1+|\eta|)^p, \tag{3.9}$$

$$a_0^{(s)}(x,\eta)\eta\geqslant 2^{p-p_2}c_4(|\eta|^p-1). \tag{3.10}$$

In view of inequalities (3.7) and (3.9), the following assertions hold: if $s\in\mathbb{N}$, $i\in\{1,\dots,n\}$, and $v\in W^{1,p}(\Omega_s)$, then $a_i^{(s)}(x,\nabla v)\in L^{p/(p-1)}(\Omega_s)$; if $s\in\mathbb{N}$ and $v\in W^{1,p}(\Omega_s)$, then $a_0^{(s)}(x,v)\in L^{p/(p-1)}(\Omega_s)$.

Now, for every $s\in\mathbb{N}$, let $\mathcal{A}_s\colon W^{1,p}(\Omega_s)\to(W^{1,p}(\Omega_s))^*$ be the operator such that for every $v,w\in W^{1,p}(\Omega_s)$,

$$\langle\mathcal{A}_s v,w\rangle=\int\limits_{\Omega_s}\Big\{\sum_{i=1}^{n}a_i^{(s)}(x,\nabla v)D_iw+a_0^{(s)}(x,v)w\Big\}dx.$$

In what follows, we denote by c_i, $i=5,6,\dots$, positive numbers depending only on n, p, p_2, c_1, c_2, c_3, c_4, and $\operatorname{meas}\Omega$.

By inequalities (3.7)–(3.10), for every $s\in\mathbb{N}$ and every $v\in W^{1,p}(\Omega_s)$, we have

$$\|\mathcal{A}_s v\|_{(W^{1,p}(\Omega_s))^*}\leqslant c_5(1+\|v\|_{W^{1,p}(\Omega_s)})^{p-1}, \tag{3.11}$$

$$\langle\mathcal{A}_s v,v\rangle\geqslant c_6\|v\|^p_{W^{1,p}(\Omega_s)}-c_7. \tag{3.12}$$

Therefore, for every $s\in\mathbb{N}$, the operator $\mathcal{A}_s$ is bounded and coercive.

In addition to (3.11) and (3.12), by inequalities (3.2), (3.3), (3.5), and (3.6), for every $s\in\mathbb{N}$ and every $v,w\in W^{1,p}(\Omega_s)$, we have

$$\|\mathcal{A}_s v-\mathcal{A}_s w\|^{p/(p-1)}_{(W^{1,p}(\Omega_s))^*}\leqslant c_8(1+\|v\|_{W^{1,p}(\Omega_s)}+\|w\|_{W^{1,p}(\Omega_s)})^{p-p_1}\|v-w\|^{p_1}_{W^{1,p}(\Omega_s)}, \tag{3.13}$$

$$\langle\mathcal{A}_s v-\mathcal{A}_s w,v-w\rangle\geqslant c_9(1+\|v\|_{W^{1,p}(\Omega_s)}+\|w\|_{W^{1,p}(\Omega_s)})^{p-p_2}\|v-w\|^{p_2}_{W^{1,p}(\Omega_s)}. \tag{3.14}$$

Therefore, for every $s\in\mathbb{N}$, the operator $\mathcal{A}_s$ is continuous and strictly monotone.

The specified properties of the operators $\mathcal{A}_s$ and the known results on the solvability of operator equations (see, e.g., [1, Chapter 2]) imply that for every $s\in\mathbb{N}$, the operator $\mathcal{A}_s$ is invertible.

We also note that if $\mathcal{A}\colon W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$ is an invertible operator and the sequence $\{\mathcal{A}_s\}$ G-converges to the operator $\mathcal{A}$, then the operator $\mathcal{A}$ is strictly monotone (in this connection, see Proposition 2.13 in [11]).

Proposition 1. *Let* $\mathcal{A}\colon W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ *be an invertible operator, and assume that the sequence* $\{\mathcal{A}_s\}$ *G-converges to the operator* $\mathcal{A}$. *Let* $v \in W^{1,p}(\Omega)$, *let* $\{g_s\} \in \mathcal{H}^*$, *and let the sequence* $\{g_s\}$ *converge strongly to the functional* $\mathcal{A}v$. *Then the following assertions hold:*

(a) *there exists a bounded sequence* $\{w_s\} \in \mathcal{H}$ *such that* $\|w_s - \mathcal{A}_s^{-1}g_s\|_{W^{1,p}(\Omega_s)} \to 0$ *and for every* $s \in \mathbb{N}$, $w_s \leqslant q_s v$ *in* Ω_s;

(b) *there exists a bounded sequence* $\{y_s\} \in \mathcal{H}$ *such that* $\|y_s - \mathcal{A}_s^{-1}g_s\|_{W^{1,p}(\Omega_s)} \to 0$ *and for every* $s \in \mathbb{N}$, $y_s \geqslant q_s v$ *in* Ω_s.

Proof. For every $s \in \mathbb{N}$, we define $v_s = \mathcal{A}_s^{-1}g_s$. Obviously, $\{v_s\} \in \mathcal{H}$. Moreover, since the sequence $\{\mathcal{A}_s\}$ G-converges to the operator $\mathcal{A}$, the sequence $\{v_s\}$ converges weakly to the function v. This means that the sequence $\{v_s\}$ is bounded and

$$\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0. \tag{3.15}$$

For every $s \in \mathbb{N}$, we define

$$\lambda_s = \big(\|v_s - q_s v\|_{L^1(\Omega_s)} + 1/s\big)^{1/2}.$$

In view of (3.15), we have

$$\lambda_s \to 0. \tag{3.16}$$

Next, for every $s \in \mathbb{N}$, we define

$$w_s = \min\{v_s - \lambda_s, q_s v\}, \qquad y_s = \max\{v_s + \lambda_s, q_s v\},$$
$$E'_s = \{v_s - \lambda_s \geqslant q_s v\}, \qquad E''_s = \{v_s + \lambda_s \leqslant q_s v\}.$$

It is clear that $\{w_s\} \in \mathcal{H}$ and $\{y_s\} \in \mathcal{H}$. Moreover, since the sequence $\{v_s\}$ is bounded, the sequences $\{w_s\}$ and $\{y_s\}$ are also bounded.

Let $s \in \mathbb{N}$. By the definition of the functions w_s and y_s, we have

$$\|w_s - v_s\|_{L^p(\Omega_s)} \leqslant \|v_s - q_s v\|_{L^p(\Omega_s)} + \lambda_s(\operatorname{meas}\Omega)^{1/p},$$
$$\|y_s - v_s\|_{L^p(\Omega_s)} \leqslant \|v_s - q_s v\|_{L^p(\Omega_s)} + \lambda_s(\operatorname{meas}\Omega)^{1/p}.$$

In addition, if $x \in E'_s$, we have $\lambda_s \leqslant (v_s - q_s v)(x)$. Therefore,

$$\lambda_s \operatorname{meas} E'_s \leqslant \|v_s - q_s v\|_{L^1(\Omega_s)} \leqslant \lambda_s^2.$$

Hence, $\operatorname{meas} E'_s \leqslant \lambda_s$. Similarly, we find that $\operatorname{meas} E''_s \leqslant \lambda_s$.

The considerations in the previous paragraph along with relations (3.15) and (3.16) impy that

$$\|w_s - v_s\|_{L^p(\Omega_s)} \to 0, \qquad \|y_s - v_s\|_{L^p(\Omega_s)} \to 0, \tag{3.17}$$

$$\operatorname{meas} E'_s \to 0, \qquad \operatorname{meas} E''_s \to 0. \tag{3.18}$$

We now show that

$$\int\limits_{E'_s} |\nabla v_s|^p \, dx \to 0. \tag{3.19}$$

We denote by z the zero function on Ω and for every $s \in \mathbb{N}$, we define $z_s = \max\{v_s - q_s v, \lambda_s\}$. Obviously, $\{z_s\} \in \mathcal{H}$. In addition, since the sequence $\{v_s\}$ is bounded, the sequence $\{z_s\}$ is also bounded. Moreover, by (3.15) and (3.16), we have

$$\|z_s\|_{L^p(\Omega_s)} \to 0. \tag{3.20}$$

Thus, the sequence $\{z_s\}$ converges weakly to the function z. Then, taking into account that the sequence $\{g_s\}$ converges strongly to the functional $\mathcal{A}v$, we conclude that $\langle g_s, z_s\rangle \to \langle \mathcal{A}v, z\rangle$. Hence, in view of the definition of the functions v_s, we get

$$\langle \mathcal{A}_s v_s, z_s\rangle \to 0. \tag{3.21}$$

We also note that

$$\int\limits_{E'_s} \Big\{ \sum_{i=1}^{n} a_i^{(s)}(x, \nabla v_s) D_i(q_s v) \Big\} dx \to 0, \tag{3.22}$$

$$\int\limits_{\Omega_s} a_0^{(s)}(x, v_s) z_s dx \to 0. \tag{3.23}$$

These relations are established using the boundedness of the sequence $\{v_s\}$, inequalities (3.7) and (3.9), the first relation in (3.18), and relation (3.20). Next, let $s \in \mathbb{N}$. By the definition of the function z_s and the set E'_s, for every $i \in \{1, \dots, n\}$, we have

$$D_i z_s = (D_i v_s - D_i(q_s v)) \cdot 1_{E'_s} \ \text{ a.e. in } \ \Omega_s.$$

Therefore, using the definition of the operator $\mathcal{A}_s$, we find that

$$\langle \mathcal{A}_s v_s, z_s\rangle = \int\limits_{E'_s} \Big\{ \sum_{i=1}^{n} a_i^{(s)}(x, \nabla v_s)(D_i v_s - D_i(q_s v)) \Big\} dx + \int\limits_{\Omega_s} a_0^{(s)}(x, v_s) z_s dx.$$

Hence, using (3.8), we obtain

$$2^{p-p_2} c_2 \int\limits_{E'_s} |\nabla v_s|^p \, dx \leqslant \int\limits_{E'_s} \Big\{ \sum_{i=1}^{n} a_i^{(s)}(x, \nabla v_s) D_i(q_s v) \Big\} dx - \int\limits_{\Omega_s} a_0^{(s)}(x, v_s) z_s dx$$
$$+ 2^{p-p_2} c_2 \operatorname{meas} E'_s + \langle \mathcal{A}_s v_s, z_s\rangle.$$

This along with the first relation in (3.18) and relations (3.21)–(3.23) implies relation (3.19).

Next, using the definition of the functions w_s and the sets E'_s, we find that for every $s \in \mathbb{N}$,

$$\sum_{i=1}^{n} \int\limits_{\Omega_s} |D_i(w_s - v_s)|^p \, dx \leqslant 2^p n \int\limits_{E'_s} |\nabla v_s|^p \, dx + 2^p n \int\limits_{E'_s} |\nabla v|^p \, dx.$$

This along with the first relation in (3.18) and relation (3.19) implies that

$$\sum_{i=1}^{n} \int\limits_{\Omega_s} |D_i(w_s - v_s)|^p \, dx \to 0.$$

Then taking into account the first relation in (3.17) and the definition of the functions v_s, we get $\|w_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0$. Moreover, as seen from the definition of the functions w_s, for every $s \in \mathbb{N}$, $w_s \leqslant q_s v$ in Ω_s. Thus, assertion (a) holds.

Next, for every $s \in \mathbb{N}$, we define $\bar{z}_s = \min\{v_s - q_s v, -\lambda_s\}$. Obviously, $\{\bar{z}_s\} \in \mathcal{H}$. In addition, since the sequence $\{v_s\}$ is bounded, the sequence $\{\bar{z}_s\}$ is also bounded. Moreover, by (3.15) and (3.16), we have

$$\|\bar{z}_s\|_{L^p(\Omega_s)} \to 0. \tag{3.24}$$

Thus, the sequence $\{\bar{z}_s\}$ converges weakly to the function z. Then, taking into account that the sequence $\{g_s\}$ converges strongly to the functional $\mathcal{A}v$, we conclude that $\langle g_s, \bar{z}_s\rangle \to \langle \mathcal{A}v, z\rangle$. Hence, in view of the definition of the functions v_s, we get

$$\langle \mathcal{A}_s v_s, \bar{z}_s\rangle \to 0. \tag{3.25}$$

Then, similarly to the proof of relation (3.19), we establish that

$$\int\limits_{E''_s} |\nabla v_s|^p \, dx \to 0. \tag{3.26}$$

In so doing, we use the definition of the functions $\bar{z}_s$ and the sets E''_s, the boundedness of the sequence $\{v_s\}$, inequalities (3.7)–(3.9), the second relation in (3.18), and relations (3.24) and (3.25). Moreover, using the definition of the functions y_s and the sets E''_s, we find that for every $s \in \mathbb{N}$,

$$\sum_{i=1}^{n} \int\limits_{\Omega_s} |D_i(y_s - v_s)|^p \, dx \leqslant 2^p n \int\limits_{E''_s} |\nabla v_s|^p \, dx + 2^p n \int\limits_{E''_s} |\nabla v|^p \, dx.$$

This along with the second relation in (3.18) and relation (3.26) implies that

$$\sum_{i=1}^{n} \int\limits_{\Omega_s} |D_i(y_s - v_s)|^p \, dx \to 0.$$

Then taking into account the second relation in (3.17) and the definition of the functions v_s, we get $\|y_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0$. In addition, as seen from the definition of the functions y_s, for every $s \in \mathbb{N}$, $y_s \geqslant q_s v$ in Ω_s. Thus, assertion (b) holds. □

4. Constraint sets and their properties

In this section, we introduce and examine the constraint sets involved in the variational inequalities studied in the present paper. We begin with the conditions on the mappings defining these sets.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be a convex function, and assume that the function h is nondecreasing or nonincreasing. Let $\varphi\colon \Omega \to \mathbb{R}$ be a function, and for every $s \in \mathbb{N}$, let $\varphi_s\colon \Omega_s \to \mathbb{R}$ be a function. Let $\{\tau_s\} \subset [0, +\infty)$, $\tau_s \to 0$, and for every $s \in \mathbb{N}$, let $\alpha_s\colon \Omega \to \mathbb{R}$ be a nonnegative function. We assume that

$$\alpha_s \to 0 \text{ a.e. in } \Omega, \tag{4.1}$$

$$\forall s \in \mathbb{N} \quad \varphi - \tau_s \leqslant \varphi_s \leqslant \varphi + \alpha_s \text{ a.e. in } \Omega_s. \tag{4.2}$$

Let $\Phi\colon W^{1,p}(\Omega) \to \mathbb{R}$ be a functional, and for every $s \in \mathbb{N}$, let $\Phi_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be a continuous convex functional. We assume that the following condition is satisfied:

- (A$_1$) for every function $v \in W^{1,p}(\Omega)$ and every sequence $\{v_s\} \in \mathcal{H}$ converging weakly to the function v, we have $\Phi_s(v_s) \to \Phi(v)$.

We note that by the convexity of the functionals Φ_s and condition (A$_1$), the functional Φ is convex.

For the further purposes, we also need some conditions combining the above functions and functionals in suitable expressions. Namely, we assume that the following conditions are satisfied:

- (A$_2$) there exist $\psi \in W^{1,p}(\Omega)$ and $c > 0$ such that $h(\psi) + \Phi(\psi) \leqslant \varphi - c$ a.e. in Ω;
- (A$_3$) there exists a bounded sequence $\{\bar{\varphi}_s\} \in \mathcal{H}$ such that for every $s \in \mathbb{N}$, $h(\bar{\varphi}_s) + \Phi_s(\bar{\varphi}_s) \leqslant \varphi_s$ a.e. in Ω_s.

Now, for every $s \in \mathbb{N}$, we define

$$V_s = \{v \in W^{1,p}(\Omega_s) : h(v) + \Phi_s(v) \leqslant \varphi_s \text{ a.e. in } \Omega_s\}.$$

Condition (A$_3$) along with the continuity and the convexity of the function h and the functionals Φ_s implies that for every $s \in \mathbb{N}$, the set V_s is nonempty, closed, and convex.

We define

$$V = \{v \in W^{1,p}(\Omega) : h(v) + \Phi(v) \leqslant \varphi \text{ a.e. in } \Omega\}.$$

Condition (A_2) along with the convexity of the function h and the functional Φ implies that the set V is nonempty and convex.

Proposition 2. *Let $\{v_s\} \in \mathcal{H}$ be a bounded sequence, and for every $s \in \mathbb{N}$, let $v_s \in V_s$. Let $\{s_j\}$ be an increasing sequence in $\mathbb{N}$, let $v \in W^{1,p}(\Omega)$, and let*

$$\|v_{s_j} - q_{s_j} v\|_{L^p(\Omega_{s_j})} \to 0. \tag{4.3}$$

Then $v \in V$.

Proof. We define the sequence $\{\bar{v}_s\}$ as follows:

$$\bar{v}_s = \begin{cases} v_s & \text{if } \ s = s_j \ \text{ for some } j \in \mathbb{N}, \\ q_s v & \text{if } \ s \neq s_j \ \text{ for any } j \in \mathbb{N}. \end{cases}$$

Obviously, $\{\bar{v}_s\} \in \mathcal{H}$ and the sequence $\{\bar{v}_s\}$ is bounded. In addition, in view of (4.3), we have $\|\bar{v}_s - q_s v\|_{L^p(\Omega_s)} \to 0$. Thus, the sequence $\{\bar{v}_s\}$ converges weakly to the function v. Then, by condition (A_1), we have $\Phi_s(\bar{v}_s) \to \Phi(v)$. Hence,

$$\Phi_{s_j}(v_{s_j}) \to \Phi(v). \tag{4.4}$$

Next, in view of the inclusions $v_s \in V_s$, there exists a set $E_1 \subset \Omega$ of measure zero such that

$$s \in \mathbb{N},\ x \in \Omega_s \setminus E_1 \Longrightarrow h(v_s(x)) + \Phi_s(v_s) \leqslant \varphi_s(x). \tag{4.5}$$

In addition, by (4.2), there exists a set $E_2 \subset \Omega$ of measure zero such that

$$s \in \mathbb{N},\ x \in \Omega_s \setminus E_2 \Longrightarrow \varphi_s(x) \leqslant \varphi(x) + \alpha_s(x). \tag{4.6}$$

We fix a sequence $\{l_s\} \in \mathcal{L}$. Obviously, for every $s \in \mathbb{N}$, we have $q_s(l_s v_s) = v_s$ a.e. in Ω_s. Then there exists a set $E_3 \subset \Omega$ of measure zero such that

$$s \in \mathbb{N},\ x \in \Omega_s \setminus E_3 \Longrightarrow (l_s v_s)(x) = v_s(x). \tag{4.7}$$

Since the sequence $\{\bar{v}_s\}$ converges weakly to the function v, by Proposition 2.8 in [11], we have $l_s\bar{v}_s \to v$ weakly in $W^{1,p}(\Omega)$. Hence, $l_{s_j} v_{s_j} \to v$ weakly in $W^{1,p}(\Omega)$. This along with condition (C_1) implies that $l_{s_j} v_{s_j} \to v$ strongly in $L^p(\Omega)$. Therefore, there exists an increasing sequence $\{t_k\} \subset \{s_j\}$ such that $l_{t_k} v_{t_k} \to v$ a.e. in Ω. In addition, by (4.1), we have $\alpha_{t_k} \to 0$ a.e. in Ω. Then there exists a set $E_4 \subset \Omega$ of measure zero such that for every $x \in \Omega \setminus E_4$, we have

$$(l_{t_k} v_{t_k})(x) \to v(x), \quad \alpha_{t_k}(x) \to 0. \tag{4.8}$$

We also note that by (4.4),

$$\Phi_{t_k}(v_{t_k}) \to \Phi(v). \tag{4.9}$$

For every $r \in \mathbb{N}$, we define

$$E^{(r)} = \Omega \setminus \bigcup_{k=r}^{\infty} \Omega_{t_k},$$

and let E_5 be the union of all the sets $E^{(r)}$, $r \in \mathbb{N}$. By condition (C_2), we have $\operatorname{meas} E_5 = 0$. We now fix $x \in \Omega \setminus (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5)$ and take an arbitrary $\varepsilon > 0$. Since $x \in \Omega \setminus E_4$, by (4.8) and the continuity of the function h, we have $h((l_{t_k} v_{t_k})(x)) \to h(v(x))$ and $\alpha_{t_k}(x) \to 0$. Then, taking into account (4.9), we find that there exists $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, $k \geqslant m$,

$$h(v(x)) \leqslant h((l_{t_k} v_{t_k})(x)) + \varepsilon, \quad \alpha_{t_k}(x) \leqslant \varepsilon, \quad \Phi(v) \leqslant \Phi_{t_k}(v_{t_k}) + \varepsilon. \tag{4.10}$$

Since $x \notin E_5$, we have $x \notin E^{(m)}$. Consequently, there exists $k \in \mathbb{N}$, $k \geqslant m$, such that $x \in \Omega_{t_k}$. Hence, $x \in \Omega_{t_k} \setminus (E_1 \cup E_2 \cup E_3)$. Then, by (4.5)–(4.7), we have

$$h(v_{t_k}(x)) + \Phi_{t_k}(v_{t_k}) \leqslant \varphi_{t_k}(x), \quad \varphi_{t_k}(x) \leqslant \varphi(x) + \alpha_{t_k}(x), \quad (l_{t_k} v_{t_k})(x) = v_{t_k}(x).$$

From this and (4.10), we deduce that $h(v(x)) + \Phi(v) \leqslant \varphi(x) + 3\varepsilon$. Therefore, in view of the arbitrariness of $\varepsilon > 0$, we have $h(v(x)) + \Phi(v) \leqslant \varphi(x)$. Hence, $h(v) + \Phi(v) \leqslant \varphi$ a.e. in Ω. Consequently, $v \in V$. ☐

Proposition 3. *Let $\mathcal{A} \colon W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be an invertible operator, and assume that the sequence $\{\mathcal{A}_s\}$ G-converges to the operator $\mathcal{A}$. Let $v \in V$, let $\{g_s\} \in \mathcal{H}^*$, and let the sequence $\{g_s\}$ converge strongly to the functional $\mathcal{A}v$. Then there exists a sequence $\{z_s\} \in \mathcal{H}$ such that $\|z_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0$ and for every $s \in \mathbb{N}$, $z_s \in V_s$.*

Proof. We first note that by (4.2), there exists a set $E' \subset \Omega$ of measure zero such that

$$s \in \mathbb{N},\ x \in \Omega_s \setminus E' \Longrightarrow \varphi(x) \leqslant \varphi_s(x) + \tau_s. \tag{4.11}$$

Since $v \in V$, we have $h(v) + \Phi(v) \leqslant \varphi$ a.e. in Ω. Then there exists a set $E'' \subset \Omega$ of measure zero such that

$$\forall x \in \Omega \setminus E'' \quad h(v(x)) + \Phi(v) \leqslant \varphi(x). \tag{4.12}$$

Moreover, by condition (A$_2$), there exist $\psi \in W^{1,p}(\Omega)$ and $c > 0$ such that $h(\psi) + \Phi(\psi) \leqslant \varphi - c$ a.e. in Ω. Then there exists a set $E''' \subset \Omega$ of measure zero such that

$$\forall x \in \Omega \setminus E''' \quad h(\psi(x)) + \Phi(\psi) \leqslant \varphi(x) - c. \tag{4.13}$$

Since the sequence $\{q_s \psi\}$ converges weakly to the function ψ, by condition (A$_1$), we have

$$\Phi_s(q_s \psi) \to \Phi(\psi). \tag{4.14}$$

Next, by Proposition 1, there exists a bounded sequence $\{w_s\} \in \mathcal{H}$ such that

$$\|w_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0, \tag{4.15}$$

$$\forall s \in \mathbb{N} \quad w_s \leqslant q_s v \ \text{ in } \ \Omega_s. \tag{4.16}$$

In addition, by the same proposition, there exists a bounded sequence $\{y_s\} \in \mathcal{H}$ such that

$$\|y_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0, \tag{4.17}$$

$$\forall s \in \mathbb{N} \quad y_s \geqslant q_s v \ \text{ in } \ \Omega_s. \tag{4.18}$$

Since the sequence $\{\mathcal{A}_s\}$ G-converges to the operator $\mathcal{A}$, the sequence $\{\mathcal{A}_s^{-1} g_s\}$ converges weakly to the function v. This along with relations (4.15) and (4.17) implies that the sequences $\{w_s\}$ and $\{y_s\}$ converge weakly to the function v. Therefore, by condition (A$_1$), we have

$$\Phi_s(w_s) \to \Phi(v), \quad \Phi_s(y_s) \to \Phi(v). \tag{4.19}$$

Since the function h is nondecreasing or nonincreasing, using relations (4.15)–(4.19), we find that there exists a bounded sequence $\{\bar{z}_s\} \in \mathcal{H}$ such that

$$\|\bar{z}_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0, \tag{4.20}$$

$$\Phi_s(\bar{z}_s) \to \Phi(v), \tag{4.21}$$

$$s \in \mathbb{N},\ x \in \Omega_s \Longrightarrow h(\bar{z}_s(x)) \leqslant h(v(x)). \tag{4.22}$$

Next, for every $s \in \mathbb{N}$, we define

$$t_s = 2(\tau_s + |\Phi_s(q_s\psi) - \Phi(\psi)| + |\Phi_s(\bar{z}_s) - \Phi(v)|).$$

We recall that by assumption, $\tau_s \to 0$. This along with relations (4.14) and (4.21) implies that $t_s \to 0$. Therefore, there exists $s_0 \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s > s_0$, we have $t_s \leqslant c$. For every $s \in \mathbb{N}$, $s > s_0$, we define

$$z_s = \frac{c}{c + t_s}\bar{z}_s + \frac{t_s}{c + t_s}q_s\psi. \tag{4.23}$$

We also recall that by condition (A$_3$), there exists a bounded sequence $\{\bar{\varphi}_s\} \in \mathcal{H}$ such that for every $s \in \mathbb{N}$, $\bar{\varphi}_s \in V_s$. For every $s \in \mathbb{N}$, $s \leqslant s_0$, we define $z_s = \bar{\varphi}_s$. Thus, we have $\{z_s\} \in \mathcal{H}$. In addition, using relation (4.20), the convergence $t_s \to 0$, and the fact that the sequence $\{\mathcal{A}_s^{-1}g_s\}$ is bounded, we find that $\|z_s - \mathcal{A}_s^{-1}g_s\|_{W^{1,p}(\Omega_s)} \to 0$. Finally, we show that for every $s \in \mathbb{N}$, $z_s \in V_s$. Let $s \in \mathbb{N}$. Obviously, if $s \leqslant s_0$, we have $z_s \in V_s$. Now, let $s > s_0$. Since the functional Φ_s is convex, by (4.23), we have

$$\Phi_s(z_s) \leqslant \frac{c}{c + t_s}\Phi_s(\bar{z}_s) + \frac{t_s}{c + t_s}\Phi_s(q_s\psi).$$

Then, taking into account the definition of t_s, we obtain

$$\Phi_s(z_s) \leqslant \frac{c}{c + t_s}\Phi(v) + \frac{t_s}{c + t_s}\Phi(\psi) + \frac{t_s}{2} - \tau_s. \tag{4.24}$$

We fix $x \in \Omega_s \setminus (E' \cup E'' \cup E''')$. Using the convexity of the function h, equality (4.23), and implication (4.22), we find that

$$h(z_s(x)) \leqslant \frac{c}{c + t_s}h(v(x)) + \frac{t_s}{c + t_s}h(\psi(x)). \tag{4.25}$$

From (4.24) and (4.25), we deduce that

$$h(z_s(x)) + \Phi_s(z_s) \leqslant \frac{c}{c + t_s}(h(v(x)) + \Phi(v)) + \frac{t_s}{c + t_s}(h(\psi(x)) + \Phi(\psi)) + \frac{t_s}{2} - \tau_s.$$

This along with (4.11)–(4.13) implies that

$$h(z_s(x)) + \Phi_s(z_s) \leqslant \varphi_s(x) - \frac{t_s(c - t_s)}{2(c + t_s)}.$$

Hence, taking into account that $t_s \leqslant c$, we get $h(z_s(x)) + \Phi_s(z_s) \leqslant \varphi_s(x)$. Thus, $h(z_s) + \Phi_s(z_s) \leqslant \varphi_s$ a.e. in Ω_s. Consequently, $z_s \in V_s$. Thus, for every $s \in \mathbb{N}$, $z_s \in V_s$. This completes the proof of the proposition. □

Remark 1. We used earlier the above conditions on the mappings defining the sets V_s in the study of the convergence of minimizers and minimum values of variational problems on these sets (see our paper [20]).

5. Convergence of solutions of variational inequalities

In this section, we give the main result of the present paper on the convergence of solutions of variational inequalities with the operators $\mathcal{A}_s$ and the constraint sets V_s.

We first note that if $s \in \mathbb{N}$ and $f \in (W^{1,p}(\Omega_s))^*$, then there exists a unique function $w \in V_s$ such that for every $v \in V_s$, we have $\langle \mathcal{A}_s w - f, w - v\rangle \leqslant 0$. This follows from the above properties of the operators $\mathcal{A}_s$ and the sets V_s and the known results on the solvability of variational inequalities (see, e.g., [1, Chapter 2]).

Theorem 1. *Let $\mathcal{A}\colon W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be an invertible operator, and assume that the sequence $\{\mathcal{A}_s\}$ G-converges to the operator $\mathcal{A}$. Let $\{f_s\} \in \mathcal{H}^*$, $f \in (W^{1,p}(\Omega))^*$, and let the sequence $\{f_s\}$ converge strongly to the functional f. Finally, for every $s \in \mathbb{N}$, let $u_s \in V_s$ and*

$$\forall v \in V_s \quad \langle \mathcal{A}_s u_s - f_s, u_s - v\rangle \leqslant 0. \tag{5.1}$$

Then there exists a function $u \in V$ such that the following assertions hold:

(a) $\forall v \in V \quad \langle \mathcal{A}u - f, u - v\rangle \leqslant 0$;

(b) *the sequence $\{u_s\}$ converges weakly to the function u;*

(c) *the sequence $\{\mathcal{A}_s u_s\}$ converges strongly to the functional $\mathcal{A}u$;*

(d) $\langle \mathcal{A}_s u_s, u_s\rangle \to \langle \mathcal{A}u, u\rangle$;

(e) *if $\{g_s\} \in \mathcal{H}^*$ and the sequence $\{g_s\}$ converges strongly to the functional $\mathcal{A}u$, then we have $\|u_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0$.*

Proof. The conclusion of the theorem follows from the above Propositions 2 and 3 and Theorem 1.2 in [8] related to abstract operators with variable domain. However, for the convenience of the reader, we expose the direct proof showing the role of Propositions 2 and 3 in the study of the asymptotic behavior of the solutions u_s of variational inequalities (5.1).

First of all we note that since the sequence $\{f_s\}$ converges strongly to the functional f, the sequence of norms $\|f_s\|_{(W^{1,p}(\Omega_s))^*}$ is bounded. This follows from Proposition 2.10 in [11]. In addition, by condition (A_3), there exists a bounded sequence $\{\bar\varphi_s\} \in \mathcal{H}$ such that for every $s \in \mathbb{N}$, $\bar\varphi_s \in V_s$. Then by (5.1), for every $s \in \mathbb{N}$, we have $\langle \mathcal{A}_s u_s - f_s, u_s - \bar\varphi_s\rangle \leqslant 0$. Using this fact, the boundedness of the sequences of norms $\|f_s\|_{(W^{1,p}(\Omega_s))^*}$ and $\|\bar\varphi_s\|_{W^{1,p}(\Omega_s)}$, and inequalities (3.11) and (3.12), we find that the sequence $\{u_s\}$ is bounded.

Next, we prove the following assertion:

(B) if $u \in V$, $\{s_j\}$ is an increasing sequence in $\mathbb{N}$, and $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, then for every $v \in V$, we have $\langle \mathcal{A}u - f, u - v\rangle \leqslant 0$.

Let $u \in V$, let $\{s_j\}$ be an increasing sequence in $\mathbb{N}$, and let $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$. We fix a sequence $\{g_s\} \in \mathcal{H}^*$ converging strongly to the functional $\mathcal{A}u$ and for every $s \in \mathbb{N}$, we define $v_s = \mathcal{A}_s^{-1} g_s$. By Proposition 3, there exists a sequence $\{z_s\} \in \mathcal{H}$ such that

$$\|z_s - v_s\|_{W^{1,p}(\Omega_s)} \to 0, \tag{5.2}$$

$$\forall s \in \mathbb{N}, \qquad z_s \in V_s. \tag{5.3}$$

Since the sequence $\{\mathcal{A}_s\}$ G-converges to the operator $\mathcal{A}$, the sequence $\{v_s\}$ converges weakly to the function u. Hence, the sequence $\{v_s\}$ is bounded. Then by (5.2), the sequence $\{z_s\}$ is also bounded. Using the boundedness of the sequences $\{v_s\}$ and $\{z_s\}$ along with relations (3.13) and (5.2), we find that

$$\|\mathcal{A}_s z_s - g_s\|_{(W^{1,p}(\Omega_s))^*} \to 0. \tag{5.4}$$

This and the boundedness of the sequences $\{u_s\}$ and $\{z_s\}$ imply that

$$\langle \mathcal{A}_s z_s - g_s, u_s - z_s\rangle \to 0. \tag{5.5}$$

In addition, taking into account that $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, the sequence $\{z_s\}$ converges weakly to the function u, and the sequences $\{f_s\}$ and $\{g_s\}$ converge strongly to the functionals f and $\mathcal{A}u$, respectively, we obtain

$$\langle f_{s_j} - g_{s_j}, u_{s_j} - z_{s_j}\rangle \to 0. \tag{5.6}$$

Obviously, for every $s \in \mathbb{N}$, we have

$$\langle \mathcal{A}_s u_s - \mathcal{A}_s z_s, u_s - z_s \rangle = \langle \mathcal{A}_s u_s - f_s, u_s - z_s \rangle + \langle f_s - g_s, u_s - z_s \rangle + \langle g_s - \mathcal{A}_s z_s, u_s - z_s \rangle.$$

This along with (5.1), (5.3), (5.5), and (5.6) implies that

$$\limsup_{j\to\infty} \langle \mathcal{A}_{s_j} u_{s_j} - \mathcal{A}_{s_j} z_{s_j}, u_{s_j} - z_{s_j} \rangle \leqslant 0.$$

Therefore, using (3.14) and the boundedness of the sequences $\{u_s\}$ and $\{z_s\}$, we establish that $\|u_{s_j} - z_{s_j}\|_{W^{1,p}(\Omega_{s_j})} \to 0$. Then by (3.13), we have

$$\|\mathcal{A}_{s_j} u_{s_j} - \mathcal{A}_{s_j} z_{s_j}\|_{(W^{1,p}(\Omega_{s_j}))^*} \to 0. \tag{5.7}$$

Now, let $v \in V$. Since the sequence $\{\mathcal{A}_s\}$ G-converges to the operator $\mathcal{A}$, by Proposition 3, there exists a sequence $\{y_s\} \in \mathcal{H}$ converging weakly to the function v and such that for every $s \in \mathbb{N}$, $y_s \in V_s$. Then by (5.1), we have

$$\forall s \in \mathbb{N} \quad \langle \mathcal{A}_s u_s - f_s, u_s - y_s \rangle \leqslant 0. \tag{5.8}$$

In addition, taking into account that $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, the sequence $\{y_s\}$ converges weakly to the function v, and the sequences $\{f_s\}$ and $\{g_s\}$ converge strongly to the functionals f and $\mathcal{A}u$, respectively, we get

$$\langle g_{s_j} - f_{s_j}, u_{s_j} - y_{s_j} \rangle \to \langle \mathcal{A}u - f, u - v \rangle. \tag{5.9}$$

It is clear that for every $s \in \mathbb{N}$,

$$\langle g_s - f_s, u_s - y_s \rangle = \langle \mathcal{A}_s u_s - f_s, u_s - y_s \rangle + \langle g_s - \mathcal{A}_s z_s, u_s - y_s \rangle + \langle \mathcal{A}_s z_s - \mathcal{A}_s u_s, u_s - y_s \rangle.$$

Hence, taking into account the boundedness of the sequences $\{u_s\}$ and $\{y_s\}$ and using relations (5.4) and (5.7)–(5.9), we deduce that $\langle \mathcal{A}u - f, u - v \rangle \leqslant 0$. Thus, assertion (B) is proved.

Next, since the sequence $\{u_s\}$ is bounded, by Proposition 2.7 in [11], there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in W^{1,p}(\Omega)$ such that $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$. Consequently, in view of Proposition 2, we have $u \in V$. Therefore, assertion (B) implies that

$$\forall v \in V \quad \langle \mathcal{A}u - f, u - v \rangle \leqslant 0. \tag{5.10}$$

Then reasoning by contradiction, we prove that the sequence $\{u_s\}$ converges weakly to the function u. In so doing, along with (5.10), we again use the boundedness of the sequence $\{u_s\}$, Proposition 2, and assertion (B). The strict monotonicity of the operator $\mathcal{A}$ is also taken into account.

Now, let $\{g_s\} \in \mathcal{H}^*$, and let the sequence $\{g_s\}$ converge strongly to the functional $\mathcal{A}u$. By Proposition 3, there exists a sequence $\{z_s\} \in \mathcal{H}$ such that $\|z_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0$ and for every $s \in \mathbb{N}$, $z_s \in V_s$. Arguing similarly as in the proof of assertion (B), we find that

$$\|\mathcal{A}_s z_s - g_s\|_{(W^{1,p}(\Omega_s))^*} \to 0, \tag{5.11}$$

$$\|u_s - z_s\|_{W^{1,p}(\Omega_s)} \to 0, \quad \|\mathcal{A}_s u_s - \mathcal{A}_s z_s\|_{(W^{1,p}(\Omega_s))^*} \to 0. \tag{5.12}$$

The strong convergence of the sequence $\{g_s\}$ to the functional $\mathcal{A}u$, relation (5.11), and the second relation in (5.12) imply that the sequence $\{\mathcal{A}_s u_s\}$ converges strongly to the functional $\mathcal{A}u$. Therefore, taking into account that the sequence $\{u_s\}$ converges weakly to the function u, we conclude that $\langle \mathcal{A}_s u_s, u_s \rangle \to \langle \mathcal{A}u, u \rangle$. Finally, from the convergence $\|z_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0$ and the first relation in (5.12), we deduce that $\|u_s - \mathcal{A}_s^{-1} g_s\|_{W^{1,p}(\Omega_s)} \to 0$. This completes the proof of the theorem. $\square$

6. Examples

First of all, we note that the fulfillment of conditions (C_1)–(C_3) is discussed, for instance, in [21]. Some examples of the functions $a_i^{(s)}$ satisfying conditions (3.1)–(3.6) are given in [11]. Therefore, below we focus only on the examples where the conditions stated at the beginning of Section 4 are satisfied.

Example 1. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the function such that for every $t \in \mathbb{R}$, $h(t) = t$. Obviously, the function h is convex and increasing. Let $\varphi\colon \Omega \to \mathbb{R}$ be a function, and assume that there exists a function $\bar{\varphi} \in W^{1,p}(\Omega)$ such that $\bar{\varphi} \leqslant \varphi$ a.e. in Ω. For every $s \in \mathbb{N}$, let $\alpha_s\colon \Omega \to \mathbb{R}$ be a nonnegative function. Now, for every $s \in \mathbb{N}$, we define $\varphi_s = (\varphi + \alpha_s)|_{\Omega_s}$. It is clear that the functions φ_s satisfy condition (4.2) with $\tau_s = 0$ for every $s \in \mathbb{N}$. Next, let Φ be the zero functional on $W^{1,p}(\Omega)$, and for every $s \in \mathbb{N}$, let Φ_s be the zero functional on $W^{1,p}(\Omega_s)$. Obviously, the functionals Φ_s are continuous and convex. In addition, the functionals Φ_s and Φ satisfy condition (A_1). Moreover, defining $\psi = \bar{\varphi} - 1$ and $c = 1$, we find that $h(\psi) + \Phi(\psi) \leqslant \varphi - c$ a.e. in Ω. Therefore, the functions h and φ and the functional Φ satisfy condition (A_2). Finally, for every $s \in \mathbb{N}$, we define $\bar{\varphi}_s = q_s\bar{\varphi}$. Then $\{\bar{\varphi}_s\} \in \mathcal{H}$, the sequence $\{\bar{\varphi}_s\}$ is bounded, and for every $s \in \mathbb{N}$, we have $h(\bar{\varphi}_s) + \Phi_s(\bar{\varphi}_s) \leqslant \varphi_s$ a.e. in Ω_s. Therefore, the functions h and φ_s and the functionals Φ_s satisfy condition (A_3). Thus, if $\alpha_s \to 0$ a.e. in Ω, then all the conditions stated at the beginning of Section 4 are satisfied. As seen, for the functions and functionals considered in this example, the sets V_s and V take the form

$$V_s = \{v \in W^{1,p}(\Omega_s) : v \leqslant \varphi_s \text{ a.e. in } \Omega_s\}, \quad V = \{v \in W^{1,p}(\Omega) : v \leqslant \varphi \text{ a.e. in } \Omega\}.$$

Example 2. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the function such that for every $t \in \mathbb{R}$, $h(t) = -t$. Obviously, the function h is convex and decreasing. Let φ, $\bar{\varphi}$, α_s, φ_s, Φ, and Φ_s be the same as in Example 1. Defining $\psi = -\bar{\varphi} + 1$ and $c = 1$, we find that $h(\psi) + \Phi(\psi) \leqslant \varphi - c$ a.e. in Ω. Therefore, the functions h and φ and the functional Φ satisfy condition (A_2). Now, for every $s \in \mathbb{N}$, we define $\bar{\varphi}_s = -q_s\bar{\varphi}$. Then $\{\bar{\varphi}_s\} \in \mathcal{H}$, the sequence $\{\bar{\varphi}_s\}$ is bounded, and for every $s \in \mathbb{N}$, we have $h(\bar{\varphi}_s) + \Phi_s(\bar{\varphi}_s) \leqslant \varphi_s$ a.e. in Ω_s. Therefore, the functions h and φ_s and the functionals Φ_s satisfy condition (A_3). Thus, if $\alpha_s \to 0$ a.e. in Ω, then all the conditions stated at the beginning of Section 4 are satisfied. As seen, for the functions and functionals considered in this example, the sets V_s and V take the form

$$V_s = \{v \in W^{1,p}(\Omega_s) : v + \varphi_s \geqslant 0 \text{ a.e. in } \Omega_s\}, \quad V = \{v \in W^{1,p}(\Omega) : v + \varphi \geqslant 0 \text{ a.e. in } \Omega\}.$$

Example 3. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the zero function. Let $c > 0$, and let $\varphi\colon \Omega \to \mathbb{R}$ be a function such that $\varphi \geqslant c$ a.e. in Ω. For every $s \in \mathbb{N}$, let $\alpha_s\colon \Omega \to \mathbb{R}$ be a nonnegative function. We assume that $\alpha_s \to 0$ a.e. in Ω, i.e., we require that condition (4.1) is satisfied. Now, for every $s \in \mathbb{N}$, we define $\varphi_s = (\varphi + \alpha_s)|_{\Omega_s}$. It is clear that the functions φ_s satisfy condition (4.2) with $\tau_s = 0$ for every $s \in \mathbb{N}$. Next, for every $s \in \mathbb{N}$, let $\Phi_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be the functional such that for every function $v \in W^{1,p}(\Omega_s)$, $\Phi_s(v) = \int_{\Omega_s} |v|^p\, dx$. It is easy to see that for every $s \in \mathbb{N}$, the functional Φ_s is continuous and convex. In order to deal with the limit functional for the sequence $\{\Phi_s\}$, we assume that the following condition is satisfied:

- $(*)$ there exists a nonnegative bounded measurable function b on Ω such that for every open cube $Q \subset \Omega$, we have $\operatorname{meas}(Q \cap \Omega_s) \to \int_Q b\, dx$.

By this condition, we have

$$\forall v \in L^1(\Omega) \qquad \int\limits_{\Omega_s} v\, dx \to \int\limits_{\Omega} bv\, dx. \tag{6.1}$$

Now, let $\Phi\colon W^{1,p}(\Omega) \to \mathbb{R}$ be the functional such that

$$\forall v \in W^{1,p}(\Omega) \qquad \Phi(v) = \int\limits_{\Omega} b|v|^p\, dx.$$

By (6.1), the functionals Φ_s and Φ satisfy condition (A_1). Next, let $\psi\colon\Omega\to\mathbb{R}$ be the zero function. Taking into account that $\varphi\geqslant c$ a.e. in Ω, we find that $h(\psi)+\Phi(\psi)\leqslant\varphi-c$ a.e. in Ω. Therefore, the functions h and φ and the functional Φ satisfy condition (A_2). Finally, for every $s\in\mathbb{N}$, let $\bar\varphi_s\colon\Omega_s\to\mathbb{R}$ be the zero function. Then $\{\bar\varphi_s\}\in\mathcal{H}$, the sequence $\{\bar\varphi_s\}$ is bounded, and for every $s\in\mathbb{N}$, we have $h(\bar\varphi_s)+\Phi_s(\bar\varphi_s)\leqslant\varphi_s$ a.e. in Ω_s. Therefore, the functions h and φ_s and the functionals Φ_s satisfy condition (A_3). Thus, all the conditions stated at the beginning of Section 4 are satisfied. As seen, for the functions and functionals considered in this example, the sets V_s and V take the form

$$V_s=\Bigl\{v\in W^{1,p}(\Omega_s):\int\limits_{\Omega_s}|v|^p\,dx\leqslant\varphi_s\ \text{ a.e. in }\ \Omega_s\Bigr\},$$

$$V=\Bigl\{v\in W^{1,p}(\Omega):\int\limits_{\Omega}b|v|^p\,dx\leqslant\varphi\ \text{ a.e. in }\ \Omega\Bigr\}.$$

Example 4. Let $h\colon\mathbb{R}\to\mathbb{R}$ be the function such that for every $t\in\mathbb{R}$, $h(t)=t$. Let $\varphi\colon\Omega\to\mathbb{R}$ be a nonnegative function, and let α_s, φ_s, and Φ_s be the same as in Example 3. We assume that condition $(*)$ stated in Example 3 is satisfied, and let Φ be the same functional as in Example 3. As mentioned above, the functionals Φ_s and Φ satisfy condition (A_1). Next, we fix $c>0$ such that

$$c^{p-1}\int\limits_{\Omega}b\,dx\leqslant 2^{-p},\tag{6.2}$$

and let $\psi\colon\Omega\to\mathbb{R}$ be the function such that for every $x\in\Omega$, $\psi(x)=-2c$. Using (6.2), we find that $h(\psi)+\Phi(\psi)\leqslant\varphi-c$ in Ω. Therefore, the functions h and φ and the functional Φ satisfy condition (A_2). Finally, for every $s\in\mathbb{N}$, let $\bar\varphi_s\colon\Omega_s\to\mathbb{R}$ be the zero function. Then $\{\bar\varphi_s\}\in\mathcal{H}$, the sequence $\{\bar\varphi_s\}$ is bounded, and for every $s\in\mathbb{N}$, we have $h(\bar\varphi_s)+\Phi_s(\bar\varphi_s)\leqslant\varphi_s$ in Ω_s. Therefore, the functions h and φ_s and the functionals Φ_s satisfy condition (A_3). Thus, all the conditions stated at the beginning of Section 4 are satisfied. As seen, for the functions and functionals considered in this example, the sets V_s and V take the form

$$V_s=\Bigl\{v\in W^{1,p}(\Omega_s):v+\int\limits_{\Omega_s}|v|^p\,dx\leqslant\varphi_s\ \text{ a.e. in }\ \Omega_s\Bigr\},$$

$$V=\Bigl\{v\in W^{1,p}(\Omega):v+\int\limits_{\Omega}b|v|^p\,dx\leqslant\varphi\ \text{ a.e. in }\ \Omega\Bigr\}.$$

Example 5. Let $h\colon\mathbb{R}\to\mathbb{R}$ be the function such that for every $t\in\mathbb{R}$, $h(t)=-t$. Let $\varphi\colon\Omega\to\mathbb{R}$ be a nonnegative function, and let α_s, φ_s, and Φ_s be the same as in Example 3. We assume that condition $(*)$ stated in Example 3 is satisfied, and let Φ be the same functional as in Example 3. As mentioned above, the functionals Φ_s and Φ satisfy condition (A_1). Next, we fix $c>0$ satisfying inequality (6.2), and let $\psi\colon\Omega\to\mathbb{R}$ be the function such that for every $x\in\Omega$, $\psi(x)=2c$. Using (6.2), we find that $h(\psi)+\Phi(\psi)\leqslant\varphi-c$ in Ω. Therefore, the functions h and φ and the functional Φ satisfy condition (A_2). Finally, for every $s\in\mathbb{N}$, let $\bar\varphi_s\colon\Omega_s\to\mathbb{R}$ be the zero function. Then $\{\bar\varphi_s\}\in\mathcal{H}$, the sequence $\{\bar\varphi_s\}$ is bounded, and for every $s\in\mathbb{N}$, we have $h(\bar\varphi_s)+\Phi_s(\bar\varphi_s)\leqslant\varphi_s$ in Ω_s. Therefore, the functions h and φ_s and the functionals Φ_s satisfy condition (A_3). Thus, all the conditions stated at the beginning of Section 4 are satisfied. As seen, for the functions and functionals considered in this example, the sets V_s and V take the form

$$V_s=\Bigl\{v\in W^{1,p}(\Omega_s):v+\varphi_s\geqslant\int\limits_{\Omega_s}|v|^p\,dx\ \text{ a.e. in }\ \Omega_s\Bigr\},$$

$$V=\Bigl\{v\in W^{1,p}(\Omega):v+\varphi\geqslant\int\limits_{\Omega}b|v|^p\,dx\ \text{ a.e. in }\ \Omega\Bigr\}.$$

Example 6. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the function such that for every $t \in \mathbb{R}$, $h(t) = e^t$. Obviously, the function h is convex and increasing. Let $\bar{c} > 1$, and let $\varphi\colon \Omega \to \mathbb{R}$ be a function such that $\varphi \geqslant \bar{c}$ a.e. in Ω. In addition, let α_s, φ_s, and Φ_s be the same as in Example 3. We assume that condition $(*)$ stated in Example 3 is satisfied, and let Φ be the same functional as in Example 3. As mentioned above, the functionals Φ_s and Φ satisfy condition (A_1). Next, we define $c = \bar{c} - 1$, and let $\psi\colon \Omega \to \mathbb{R}$ be the zero function. It is easy to see that $h(\psi) + \Phi(\psi) \leqslant \varphi - c$ a.e. in Ω. Therefore, the functions h and φ and the functional Φ satisfy condition (A_2). Finally, for every $s \in \mathbb{N}$, let $\bar{\varphi}_s\colon \Omega_s \to \mathbb{R}$ be the zero function. Then $\{\bar{\varphi}_s\} \in \mathcal{H}$, the sequence $\{\bar{\varphi}_s\}$ is bounded, and for every $s \in \mathbb{N}$, we have $h(\bar{\varphi}_s) + \Phi_s(\bar{\varphi}_s) \leqslant \varphi_s$ a.e. in Ω_s. Therefore, the functions h and φ_s and the functionals Φ_s satisfy condition (A_3). Thus, all the conditions stated at the beginning of Section 4 are satisfied. As seen, for the functions and functionals considered in this example, the sets V_s and V take the form

$$V_s = \Big\{ v \in W^{1,p}(\Omega_s) : e^v + \int\limits_{\Omega_s} |v|^p\,dx \leqslant \varphi_s \ \text{ a.e. in } \ \Omega_s \Big\},$$

$$V = \Big\{ v \in W^{1,p}(\Omega) : e^v + \int\limits_{\Omega} b|v|^p\,dx \leqslant \varphi \ \text{ a.e. in } \ \Omega \Big\}.$$

Example 7. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the function such that for every $t \in \mathbb{R}$, $h(t) = -t$. Let $\varphi\colon \Omega \to \mathbb{R}$ be the zero function, and for every $s \in \mathbb{N}$, let $\varphi_s\colon \Omega_s \to \mathbb{R}$ be the zero function. Let $\{\Phi_s\} \in \mathcal{H}^*$, and let $\Phi \in (W^{1,p}(\Omega))^*$. We assume that the sequence $\{\Phi_s\}$ converges strongly to the functional Φ. Obviously, the functionals Φ_s and Φ satisfy condition (A_1). Next, let $\psi_0\colon \Omega \to \mathbb{R}$ be the function such that for every $x \in \Omega$, $\psi_0(x) = 1$. It is clear that $\psi_0 \in W^{1,p}(\Omega)$. We assume that $\langle \Phi, \psi_0 \rangle \neq 1$. If $\langle \Phi, \psi_0 \rangle < 1$, then setting $\psi = \psi_0$ and $c = 1 - \langle \Phi, \psi_0 \rangle$, we obtain the equality $h(\psi) + \Phi(\psi) = \varphi - c$. We get the same equality also in the case $\langle \Phi, \psi_0 \rangle > 1$ by setting $\psi = -\psi_0$ and $c = \langle \Phi, \psi_0 \rangle - 1$. Therefore, the functions h and φ and the functional Φ satisfy condition (A_2). Finally, for every $s \in \mathbb{N}$, let $\bar{\varphi}_s\colon \Omega_s \to \mathbb{R}$ be the zero function. Then $\{\bar{\varphi}_s\} \in \mathcal{H}$, the sequence $\{\bar{\varphi}_s\}$ is bounded, and for every $s \in \mathbb{N}$, we have $h(\bar{\varphi}_s) + \Phi_s(\bar{\varphi}_s) = \varphi_s$. Therefore, the functions h and φ_s and the functionals Φ_s satisfy condition (A_3). Thus, all the conditions stated at the beginning of Section 4 are satisfied. As seen, for the functions and functionals considered in this example, the sets V_s and V take the form

$$V_s = \{ v \in W^{1,p}(\Omega_s) : v \geqslant \langle \Phi_s, v \rangle \ \text{ a.e. in } \ \Omega_s \}, \quad V = \{ v \in W^{1,p}(\Omega) : v \geqslant \langle \Phi, v \rangle \ \text{ a.e. in } \ \Omega \}.$$

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Received October 15, 2025
Revised October 28, 2025
Accepted November 3, 2025

Funding Agency: This work was partially supported by the Ministry of Science and Higher Education of the Russian Federation within the framework of the Program of development of Ural Federal University under the “Priority-2030” academic leadership program.

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Cite this article as: A. A. Kovalevsky. Nonlinear elliptic variational inequalities with unilateral pointwise functional constraints in variable domains. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2025, vol. 31, no. 4, pp. 132–148.