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OSCILLATION AND NON-OSCILLATION CRITERIA FOR SECOND ORDER LINEAR NONHOMOGENEOUS FUNCTIONAL-DIFFERENTIAL EQUATIONS

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The Riccati equation method is used to establish oscillation and non-oscillation criteria for second order linear nonhomogeneous functional-differential equations. We show that the obtained oscillation criterion is a generalization of J. S. W. Wong's oscillation criterion for second order linear nonhomogeneous ordinary differential equations. Two examples, demonstrating the aptitude of the obtained criteria, are presented.

Keywords: Riccati equations, functional-differential equations, oscillation, interval oscillation, non-oscillation.

Г. А. Григорян. Критерии колебательности и неколебательности для линейных неоднородных функционально-дифференциальных уравнений второго порядка.

Метод уравнения Риккати применяется для установления критериев осцилляционности и неосцилляционности для линейных неоднородных функционально-дифференциальных уравнений второго порядка. Показано, что полученный критерий осцилляционности является обобщением критерия осцилляционности Дж. С. В. Вонга для линейных неоднородных обыкновенных дифференциальных уравнений второго порядка. Приведены два примера, демонстрирующие применимость полученных критериев.

Ключевые слова: уравнения Риккати, функционально-дифференциальные уравнения, колебание, интервальное колебание, отсутствие колебания.

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1. Introduction

Let $p(t),\ q(t),\ f(t),\ r_j(t),\ j=\overline{1,n}$, be real-valued locally integrable, $\alpha_j(t),\ j=\overline{1,n}$, be locally measurable functions on $[t_0,\infty)$, and let $p(t)>\ 0,\ \alpha_j(t)\le t,\ j=\overline{1,n},\ t\ge t_0$. Consider the second order linear functional-differential equation

$$(p(t)\phi'(t))'+q(t)\phi'(t)+\sum_{j=1}^{n} r_j(t)\phi(\alpha_j(t))=f(t),\quad t\in\mathbb{R}. \tag{1.1}$$

Let $\theta(t)$ be a continuous function on $(-\infty,t_1]$, for some $t_1\ge t_0$ and let $\zeta\in\mathbb{R}$. By a Cauchy problem for Eq. (1.1) we mean to find a continuous on $\mathbb{R}$ function $\phi(t)$, which is continuously differentiable on $[t_1,\infty)$ with its absolutely continuous derivative on $[t_1,\infty)$, satisfies (1.1) almost everywhere on $[t_1,\infty)$ and satisfies the initial conditions $\phi(t)=\theta(t),\ t\le t_1,\ \phi'(t_1)=\ \zeta$. Equation (1.1) should be understood in the sense of its equivalence to the following linear system of functional-differential equations

$$\begin{cases} \phi'(t)=\dfrac{1}{p(t)}\psi(t),\\ \psi'(t)=-\sum\limits_{j=1}^{n} r_j(t)\phi(\alpha_j(t))-\dfrac{q(t)}{p(t)}\psi(t)+f(t),\quad t\ge t_0,\end{cases}$$

which can be realized by involving a new dependent variable $\psi(t)$ by

$$\psi(t)=p(t)\phi'(t),\quad t\ge t_0.$$

This fact, on the basis of Theorem 2.3 (see below), implies that for $\zeta \in \mathbb{R}, t_1 \geq t_0$ and real-valued continuous function $\theta(t)$ on $(-\infty, t_1]$ the Cauchy problem for Eq. (1.1) with the initial conditions $\phi(t) = \theta(t), t \leq t_1, \phi'(t_1) = \zeta$ has always a unique solution. By a solution of Eq. (1.1) we mean the solution of a Cauchy problem for this equation (for any $t_1 \geq t_0$).

Definition 1.1. A solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeroes.

Definition 1.2. Eq. (1.1) is called oscillatory if all of its solutions are oscillatory, otherwise it is called nonoscillatory.

Definition 1.3. Eq. (1.1) is called oscillatory on the interval $[a, b] \subset [t_0, \infty)$ if its every solution has a zero on $[a, b]$.

One of the most important problems in the study of linear functional differential equations, in particular of Eq. (1.1), is that of finding conditions, providing oscillatory or nonoscillatory behavior of their solutions. Many publications are devoted to this problem (see [1–3; 5; 9–12] and references therein). One of approaches in studying of this problem is that of finding explicit conditions on coefficients of the studying equation, providing its oscillation (or non-oscillation). Results in this direction have been obtained in [1; 3; 9–12]. Another approach is that of comparing the considered equation with the another (functional-differential or ordinary differential) equation. This approach allows by means of properties of solutions of relatively simple equation to describe (to detect) wide classes of oscillatory and (or) nonoscillatory equations. Results of this type have been obtained, for example, in [1; 5; 10]. Finally, an interesting approach is that of reducing the oscillation problem for functional-differential equations to the oscillation problem for ordinary differential equations. A result in this direction has been obtained in (see [2, Theorem 2]).

In this paper we use the Riccati equation method to establish oscillation and non-oscillation criteria for Eq. (1.1).

Let $d(t) > 0$, $r(t)$ and $g(t)$ be real-valued continuous functions on $[t_0, \infty)$. Consider the second order linear ordinary differential equation

$$(d(t)\phi')' + r(t)\phi = g(t), \quad t \geq t_0. \tag{1.2}$$

Combining the Riccati equation method with a variational technique in [13] J. S. W. Wong proved the following oscillation theorem.

Theorem 1.1 [8, Theorem 1]. *Suppose that for every $T \geq t_0$ there exist s_1, t_1, s_2, t_2, satisfying the conditions $T \leq s_1 < t_1 \leq s_2 < t_2$,*

$$g(t) \begin{cases} \leq 0, & t \in [s_1, t_1], \\ \geq 0, & t \in [s_2, t_2]. \end{cases}$$

Denote $D(s_i, t_i) \equiv \{u \in \mathrm{C}^1[s_i, t_i] | u(t) \not\equiv 0, u(s_i) = u(t_i) = 0\}$, $i = 1, 2$. If there exists $u \in D(s_i, t_i)$ such that

$$\int\limits_{s_i}^{t_i} (r(\tau)u(\tau)^2 - d(\tau)u'(\tau)^2)\mathrm{d}\tau \geq 0$$

for $i = 1, 2$, then Eq. (1.2) is oscillatory.

We show that the obtained below oscillation criterion is a generalization of Theorem 1.1.

2. Auxiliary Propositions

Let $a(t)$, $b(t)$, $c(t)$, $a_1(t)$, $b_1(t)$, $c_1(t)$ be real-valued locally integrable functions on $[t_0, \infty)$. Consider the Riccati equations

$$y'(t) + a(t)y^2(t) + b(t)y(t) + c(t) = 0, \tag{2.1}$$

$$y'(t) + a_1(t)y^2(t) + b_1(t)y(t) + c_1(t) = 0, \tag{2.2}$$

$t \geq t_0$ and the differential inequalities

$$\eta'(t) + a(t)\eta^2(t) + b(t)\eta(t) + c(t) \geq 0, \tag{2.3}$$

$$\eta'(t) + a_1(t)\eta^2(t) + b_1(t)\eta(t) + c_1(t) \geq 0, \tag{2.4}$$

$t \geq t_0$. By a solution of Eq. (2.1) (Eq. (2.2), inequalities (2.3), (2.4)) on $[t_1, t_2)$ ($t_0 \leq t_1 < t_2 \leq \infty$) we mean an absolutely continuous function on $[t_1, t_2)$, satisfying (2.1) ((2.2)–(2.4)) almost everywhere on $[t_1, t_2)$. Since the function $a(t)y^2 + b(t)y + c(t)$ ($a_1(t)y^2 + b_1(t)y + c_1(t)$) satisfies the Caratheodory condition on $[t_0, \infty) \times \mathbb{R}$, for every $t_1 \geq t_0$, $\gamma \in \mathbb{R}$ there exists $t_2 > t_1$ such that Eq. (2.1) (Eq. (2.2)) has a solution $y(t)$ on $[t_1, t_2)$ with $y(t_1) = \gamma$. Note that every solution of Eq. (2.1) (Eq. (2.2)) is a solution of the inequality (2.3) ((2.4)). Note also, that for $a(t) \geq 0$ ($a_1(t) \geq 0$), $t \geq t_0$, the real-valued solutions of the equation $\eta'(t) + b(t)\eta(t) + c(t) = 0$ ($\eta'(t) + b_1(t)\eta(t) + c_1(t) = 0$) are solutions of the inequality (2.3) ((2.4)). Therefore for $a(t) \geq 0$ ($a_1(t) \geq 0$), $t \geq t_0$, the inequality (2.3) ((2.4)) has a solution on $[t_0, \infty)$, satisfying any initial real-valued condition. In the sequel we will assume that the solutions of considered equations and inequalities are real-valued.

Theorem 2.1. *Let $y_0(t)$ be a solution of Eq. (2.1) on $[t_1, t_2)$, and $\eta_0(t)$, $\eta_1(t)$ be solutions of inequalities (2.3) and (2.4) with $\eta_0(t_1) \geq y_0(t_1)$, $\eta_1(t_1) \geq y_0(t_1)$ respectively, and let $a_1(t) \geq 0$,*

$$\lambda - y_0(t_1) + \int\limits_{t_1}^{t} \exp\Big\{\int\limits_{t_1}^{\tau} \big[a_1(\xi)(\eta_0(\xi) + \eta_1(\xi)) + b_1(\xi)\big] d\xi\Big\}$$

$$\times \big[(a(\tau) - a_1(\tau))y_0^2(\tau) + (b(\tau) - b_1(\tau))y_0(\tau) + c(\tau) - c_1(\tau)\big] d\tau \geq 0, \quad t \in [t_1, t_2),$$

for some $\lambda \in [y_0(t_1), \eta_1(t_1)]$. Then Eq. (2.2) has a solution $y_1(t)$ on $[t_1, t_2)$ with $y_1(t_1) \geq y_0(t_1)$, moreover $y_1(t) \geq y_0(t)$, $t \in [t_1, t_2)$.

Proof. By analogy with the proof of Theorem 3.1 from [4]. □

Consider the equation

$$y'(t) + \frac{1}{p(t)}y^2(t) + \frac{q(t)}{p(t)}y(t) - \frac{f(t)}{\lambda} \exp\Big\{-\int\limits_{t_1}^{t} \frac{y(\tau)}{p(\tau)} d\tau\Big\}$$

$$+ \sum_{j=1}^{n} r_j(\tau) \exp\Big\{-\int\limits_{\alpha_j(t)}^{t} \frac{y(\tau)}{p(\tau)} d\tau\Big\} = 0, \quad t \geq t_1 \ (\geq t_0), \quad \lambda = \text{const} \neq 0. \tag{2.5}$$

By a solution of this equation on $[t_1, t_2)$ ($t_0 \leq t_1 < t_2 \leq \infty$) we mean a continuous function $y(t)$ on $(-\infty, t_2)$ that is absolutely continuous on $[t_1, t_2)$ and satisfies (2.5) almost everywhere on $[t_1, t_2)$.

Let $t_0 \leq t_1 < t_2 \leq \infty$. Denote $T(t_1, t_2) \equiv \min\{t_1, \min\limits_{1 \leq j \leq n} \{\inf\limits_{t \in [t_1, t_2)} \alpha_j(t)\}\}$.

Let $\phi_0(t)$ be a solution of Eq. (1.1) on $[t_1, t_2)$, and let $\phi_0(t) \neq 0$, $t \in [T(t_1, t_2), t_2)$. It is easy to show that

$$y_0(t) \equiv \begin{cases} \dfrac{p(t)\phi_0'(t)}{\phi_0(t)}, & t \in [t_1, t_2), \\ \dfrac{p(t_1)\phi_0'(t_1)}{\phi_0(t_1)}, & t \leq t_1, \end{cases} \tag{2.6}$$

is a solution of Eq. (2.5) on $[t_1, t_2)$, where $\lambda = \phi_0(t_1)$.

Definition 2.1. An interval $[t_1, t_2)$, $t_0 \leq t_1 < t_2 \leq \infty$, is called the maximum existence interval for a solution $y(t)$ of Eq. (2.5) if it exists on $[t_1, t_2)$ and cannot be continued to the right from t_2 as a solution of Eq. (2.5).

Lemma 2.1. *Let $y(t)$ be a solution of Eq. (2.5) on $[t_1, t_2)$ with $t_2 < +\infty$. If the function $F(t) \equiv \int_{t_1}^{t} \frac{y(\tau)}{p(\tau)} d\tau$, $t_1 \le t < t_2$, is bounded from below, then $[t_1, t_2)$ cannot be the maximum existence interval for $y(t)$.*

Proof. By (2.6) $\phi(t) \equiv \exp\{F(t)\}$, $t_1 \le t < t_2$, is a solution to Eq. (1.1). From here and the condition of the lemma it follows that $\phi(t) \ne 0$, $t \in [t_1, t_3)$, for some $t_3 > t_2$. Then by (2.6) the function $\frac{p(t)\phi'(t)}{p(t)}$, $t_1 \le t < t_3$, is a solution of Eq (2.5) on $[t_1, t_3)$, that is a continuation of $y(t)$. Therefore, $[y_1, t_2)$ is not the maximum existence interval for $y(t)$.

The lemma is proved.

Let $r_{1,j}(t)$, $j = \overline{1,n}$, $t \ge t_0$, be real-valued locally integrable functions on $[t_0, \infty)$. Consider the equation

$$y'(t) + \frac{1}{p(t)} y^2(t) - \frac{q(t)}{p(t)} y(t) + \sum_{j=1}^{n} r_{1,j}(\tau) \exp\Big\{ - \int\limits_{\alpha_j(t)}^{t} \frac{y(\tau)}{p(\tau)} d\tau \Big\} = 0, \quad t \ge t_1 \ (\ge t_0). \tag{2.7}$$

Lemma 2.2. *Let $y_1(t)$ be a solution of Eq. (2.7) on $[t_1, t_2)$, and let the following conditions be satisfied.*

(1) $r_{1,j}(t) \ge r_j(t)$, $t \in [t_1, t_2)$, $j = \overline{1,n}$.

(2) *If $r_j(t) < 0$, then $r_{1,j}(t) \ge 0$ or $\alpha_j(t) = t$ for every $t \in [t_1, t_2)$, $j = \overline{1,n}$.*

(3) $\frac{f(t)}{\lambda} \ge 0$, $t \in [t_1, t_2)$.

Then for every real-valued continuous function $\gamma(t) \ge y_1(t)$, $t \le t_1$, $\gamma(t_1) > y_1(t_1)$, Eq. (2.5) has a solution $y(t)$ on $[t_1, t_2)$ with $y(t) = \gamma(t)$, $t \le t_1$, and

$$y(t) > y_1(t), \quad t \in [t_1, t_2). \tag{2.8}$$

Proof. Let $y(t)$ be a solution of Eq. (2.5) with $y(t) = \gamma(t)$, $t \le t_1$, and let $[t_1, t_3)$ be its maximum existence interval. Suppose $t_3 < t_2$. Show that

$$y(t) > y_1(t), \quad t \in [t_1, t_3). \tag{2.9}$$

We set

$$R(t) \equiv \sum_{j=1}^{n} r_j(\tau) \exp\Big\{ - \int\limits_{\alpha_j(t)}^{t} \frac{y(\tau)}{p(\tau)} d\tau \Big\} - \frac{f(t)}{\lambda} \exp\Big\{ - \int\limits_{t_1}^{t} \frac{y(\tau)}{p(\tau)} d\tau \Big\},$$

$$R_1(t) \equiv \sum_{j=1}^{n} r_{1,j}(\tau) \exp\Big\{ - \int\limits_{\beta_j(t)}^{t} \frac{y_1(\tau)}{p(\tau)} d\tau \Big\}, \quad t \in [t_1, t_3).$$

Suppose (2.9) is not true. Since $y(t_1) > y_1(t_1)$, there exists $t_4 \in (t_1, t_3)$ such that

$$y(t) > y_1(t), \quad t \in [t_1, t_4), \quad \text{and} \quad y(t_4) = y_1(t_4). \tag{2.10}$$

Let us show that

$$R(t) \le R_1(t), \quad t \in [t_1, t_4). \tag{2.11}$$

Denote

$$\Delta_j(t) \equiv r_{1,j}(t) \exp\Big\{ - \int\limits_{\alpha_j(t)}^{t} \frac{y_1(\tau)}{p(\tau)} d\tau \Big\} - r_j(t) \exp\Big\{ - \int\limits_{\alpha_j(t)}^{t} \frac{y(\tau)}{p(\tau)} d\tau \Big\}, \quad t \in [t_1, t_3).$$

Obviously if $r_{1,j}(t) \geq 0$, $r_j(t) < 0$, then $\Delta_j(t) \geq 0$. If $r_j(t) < 0$ and $\alpha_j(t) = t$, then by the condition (1) we have $\Delta = r_{1,j}(t) - r_j(t) \geq 0$. Assume $r_j(t) \geq 0$. Then

$$\Delta_j(t) = [r_{1,j}(t) - r_j(t)] \exp\Big\{-\int\limits_{\alpha_j(t)}^{t} \frac{y_1(\tau)}{p(\tau)}d\tau\Big\} + r_j(t) \exp\Big\{-\int\limits_{\alpha_j(t)}^{t} \frac{y(\tau)}{p(\tau)}d\tau\Big\}$$

$$\times\Big[1 - \exp\Big\{-\int\limits_{\alpha_j(t)}^{t} \frac{[y_1(\tau) - y(\tau)]}{p(\tau)}d\tau\Big\}\Big]$$

This together with the condition (1) and (2.10) implies that $\Delta_j(t) \geq 0$. Hence, (2.11) is valid.

Consider the Riccati equations

$$y' + \frac{1}{p(t)}y^2 + q(t)y + R(t) = 0, \quad t \in [t_1, t_3), \tag{2.12}$$

$$y' + \frac{1}{p(t)}y^2 + q(t)y + R_1(t) = 0, \quad t \in [t_1, t_3). \tag{2.13}$$

Note that $y(t)$ and $y_1(t)$ are solutions of the equations (2.12) and (2.13) respectively on $[t_1, t_3)$. Then since $y(t_1) > y_1(t_1)$ by Theorem 2.1 from (2.11) it follows that $y(t_4) > y_1(t_4)$, which contradicts (2.10). The obtained contradiction proves (2.9). It follows from (2.9) that the function $F(t) \equiv \int_{t_1}^{t} \frac{y(t)}{p(t)}$, $t \in [t_1, t_3)$, is bounded from below on $[t_1, t_3)$. In virtue of Lemma 2.1 from here it follows that $[t_1, t_3)$ is not the maximum existence interval for $y(t)$, which contradicts our supposition. The obtained contradiction shows that $y(t)$ exists on $[t_1, t_2)$ and the inequality (2.8) is valid. If $t_3 \geq t_2$ then the existence $y(t)$ on $[t_1, t_2)$ is clear and the inequality (2.8) follows from the already proven fact, that for any $t_5 \in (t_1, t_2)$ the inequality $y(t) > y_1(t)$, $t \in [t_1, t_5)$, is valid.

The lemma is proved.

The following statement can be justified similarly to the proof of Lemma 2.2.

Lemma 2.3. *Let $y(t)$ be a solution of Eq. (2.5) on $[t_1, t_2]$, and let the following conditions be satisfied.*

(1′) $r_{1,j}(t) \leq r_j(t)$, $t \in [t_1, t_2)$, $j = \overline{1, n}$.

(2′) *If $r_{1,j}(t) < 0$, then $r_j(t) \geq 0$ or $\alpha_j(t) = t$ for every $t \in [t_1, t_2]$, $j = \overline{1, n}$.*

(3′) $\frac{f(t)}{\lambda} \leq 0$, $t \in [t_1, t_2]$.

Then for every real-valued continuous function $\gamma(t) \geq y(t)$, $t \leq t_1$, $\gamma(t_1) > y(t_1)$, Eq. (2.7) has a solution $y_1(t)$ on $[t_1, t_2]$ with $y_1(t) = \gamma(t)$, $t \leq t_1$, and

$$y_1(t) > y(t), \quad t \in [t_1, t_2].$$

Let $\beta_j(t), j = \overline{1, n}$, be locally measurable functions on $[t_0, \infty)$ and let $t_0 \leq t_1 < t_2 \leq t_3 < t_4 < \infty$. We set: $\omega_+ \equiv \{k \in \Omega | \beta_k(t) \geq t,\ t \in [t_1, t_2]\}$; $\omega_1^- \equiv \{k \in \Omega\colon t_1 \leq \beta_k(t) \leq t, t \in [t_1, t_2]\}$; $\omega_2^- \equiv \{k \in \Omega\colon \beta_k(t) \leq t, t \in [t_3, t_4]\}$;

$$T_1 \equiv \inf_{\substack{t\in[t_1,t_4]\\ k\in\Omega}} \beta_k(t); \quad T_2 \equiv \sup_{\substack{t\in[t_1,t_4]\\ k\in\Omega}} \beta_k(t); \quad t_2^+ \equiv \sup_{\substack{t\in[t_1,t_2]\\ k\in\omega_+}} \beta_k(t); \quad t_3^- \equiv \inf_{\substack{t\in[t_3,t_4]\\ k\in\omega_2^-}} \beta_k(t).$$

Consider the equations

$$(p(t)\phi'(t))' + \sum_{k=1}^{n} r_k(t)\phi(\beta_k(t)) = f(t), \quad t \geq t_0. \tag{2.14}$$

$$(p(t)\phi'(t))' + \Bigg[\sum_{k\in\omega_+} r_k(t) \exp\Bigg\{ \int\limits_t^{\beta_k(t)} \frac{d\tau}{p(\tau)} \int\limits_\tau^{t_2} \Big(\sum_{k\in\omega_+} r_k(s) \Big) ds \Bigg\}$$
$$+ \sum_{k\in\omega_1^-} r_k(t) \frac{\displaystyle\int_{t_1}^{\beta_k(t)} \frac{d\tau}{p(\tau)} + \varepsilon}{\displaystyle\int_{t_1}^{t} \frac{d\tau}{p(\tau)} + \varepsilon} \Bigg] \phi(t) = 0, \quad t \in [t_1, t_2], \quad \varepsilon > 0; \tag{2.15}$$

$$(p(t)\phi'(t))' + \Big(\sum_{k\in\omega_2^-} r_k(t) \Big) \phi(t) = 0, \quad t \in [t_3, t_4]. \tag{2.16}$$

Theorem 2.2 [5, Theorem 2.6]. *Let $\omega_+ \cup \omega_1^- \neq \varnothing$, $\omega_2^- \neq \varnothing$ and the following conditions be satisfied*:

(a) $r_k(t) \geq 0,\ t \in [T_1, T_2],\ k \in \Omega$;

(b) $t_2^+ \leq t_3^-$;

(c) *for some* $\varepsilon_0 > 0$ *Eq.* (2.15) *is oscillatory on* $[t_1, t_2]$ *for all* $\varepsilon \in (0, \varepsilon_0)$;

(d) *Eq.* (2.16) *is oscillatory on* $[t_3, t_4]$.

Then Eq. (2.14) *is oscillatory on* $[T_1, T_2]$.

Let $a_{kjm}(t),\ b_k(t),\ k, j = \overline{1, n},\ m = \overline{1, N}$, be real-valued locally integrable functions on $[t_0, \infty)$ and let $\alpha_{kj}(t),\ t \geq t_0,\ k = \overline{1, n},\ j = \overline{1, N}$, be locally measurable functions on $[t_0, \infty)$, satisfying the retorsion condition

$$\alpha_{kj}(t) \leq t, \quad t \geq t_0, \quad k = \overline{1, n}, \quad j = \overline{1, N}.$$

Consider the linear system of functional differential equations

$$\phi_k'(t) = \sum_{j=1}^n \sum_{m=1}^N a_{kjm}(t) \phi_j(\alpha_{jm}(t)) + b_k(t), \quad t \geq t_0, \quad k = \overline{1, n}. \tag{2.17}$$

Let $r_k(t),\ k = \overline{1, n}$, be real-valued continuous functions on $(-\infty, t_0]$. By a Cauchy problem for the system (2.17) we mean to find a real-valued continuous vector function $(\phi_1(t), \dots, \phi_n(t))$ on $\mathbb{R}$ that is absolutely continuous on $[t_0, \infty)$, and that satisfies (2.17) almost everywhere on $[t_0, \infty)$ and the initial conditions

$$\phi_k(t) = r_k(t), \quad t \leq t_0, \quad k = \overline{1, n}.$$

Theorem 2.3 [7, Corollary 3.1]. *The Cauchy problem* (2.17), (2.8) *has a unique solution.*

3. Oscillation and Non-Oscillation Criteria

Along with Eq. (1.1) consider the equation

$$(p(t)\phi'(t))' + q(t)\phi'(t) + \sum_{j=1}^n r_{1,j}(t) \phi(\alpha_j(t)) = 0, \quad t \geq t_0. \tag{3.1}$$

Theorem 3.1. *Let the following conditions be satisfied.*

(1) $r_{1,j}(t) \geq r_j(t),\ t \geq t_0,\ j = \overline{1, n}$.

(2) *If* $r_j(t) < 0$, *then* $r_{1,j}(t) \geq 0$ *or* $\alpha_j(t) = t$ *for every* $t \geq t_0,\ j = \overline{1, n}$.

(3) $f(t) \geq 0,\ t \geq t_0$.

(4) $\lim\limits_{t\to\infty} \alpha_j(t) = \infty,\ j = \overline{1, n}$.

If Eq. (3.1) *is nonoscillatory, then Eq.* (1.1) *is also nonoscillatory.*

Proof. Let Eq. (3.1) be nonoscillatory. Then it has a solution $\phi_1(t)$ such that $\phi_1(t) \neq\ 0, t \geq t_1$, for some $t_1 \geq t_0$, and, therefore, the condition (4) implies that

$$\phi(\alpha_j(t)) \neq 0, \quad t \geq T, \quad j = \overline{1,n}.$$

for some $T > t_1$. By virtue of (2.6) from here it follows that Eq. (2.7) has a solution $y_1(t) \equiv \dfrac{p(t)\phi'(t)}{p(t)}$ on $[T, \infty)$. Then by Lemma 2.2 it follows from the conditions (1), (2) and (3) that Eq. (2.5) has a solution $y(t)$ on $[T, \infty)$. Hence by (2.6)

$$\phi(t) \equiv \exp\left\{\int\limits_T^t \frac{y(\tau)}{p(\tau)} d\tau\right\}, \quad t \in \mathbb{R},$$

is a nonoscillatory solution of Eq. (1.1) on $[T, \infty)$. Therefore Eq. (1.1) is nonoscillatory.

The theorem is proved.

We set $r_j^+(t) \equiv \max\{0, r_j(t)\},\ t \geq t_0,\ j = \overline{1,n}$. Consider the equation

$$(p(t)\phi'(t))' + q(t)\phi'(t) + \sum_{j=1}^n r_j^+(t)\phi(\alpha_j(t)) = 0, \quad t \geq t_0. \tag{3.2}$$

The theorem is proved.

Since $r_j^+(t) \geq 0,\ t \geq t_0,\ j = \overline{1,n}$, from Theorem 3.1 we obtain immediately

Corollary 3.1. *Let the conditions* (3) *and* (4) *of Theorem* 3.1 *be satisfied. If Eq.* (3.2) *is nonoscillatory, then Eq.* (1.1) *is also nonoscillatory.*

Example 3.1. Consider the equation

$$\phi''(t) + \sin^2 t\, \phi(\alpha_1(t)) + \cos^2 t\, \phi(\alpha_2(t)) - \phi(t) = \cos(\sin(\ln(1 + t))), \quad t \geq 0, \tag{3.3}$$

$\lim\limits_{t\to+\infty} \alpha_j(t) = +\infty,\ j = 1, 2$. For establishing non-oscillation of this equation we compare it with the following homogeneous one

$$\phi''(t) + \sin^2 t\, \phi(\alpha_1(t)) + \cos^2 t\, \phi(\alpha_2(t)) - \phi(t) = 0, \quad t \geq 0, \tag{3.4}$$

It is not difficult to verify that for $r_{1,1}(t) = r_1(t) = \sin^2 t,\ r_{1,2}(t) = r_2(t) = \cos^2 t,\ r_{1,3}(t) = r_3(t) = -1,\ f(t) = \cos(\sin(\ln(1+t))),\ t \geq 0$, the conditions (1)–(4) of Theorem 3.1 for Eqs. (3.3) and (3.4) are satisfied. Obviously, $\phi(t) \equiv 1$ is a nonoscillating solution for Eq. (3.4). Then by Theorem 3.1 Eq. (3.3) is nonoscillatory. □

Theorem 3.2. *Let the following conditions be satisfied.*

I) $r_j(t) \geq r_{1,j}(t),\ t \geq t_0,\ j = \overline{1,n}$.

II) *If* $r_{1,j}(t) < 0$, *then* $r_j(t) \geq 0$, *or* $\alpha_j(t) = t$, *for every* $t \geq t_0,\ j = \overline{1,n}$.

III) $\lim\limits_{t\to\infty} \alpha_j(t) = \infty,\ j = \overline{1,n}$.

IV) *For every* $T \geq t_0$ *there exist* s_1, t_1, s_2, t_2, *satisfying the conditions* $T < s_1 < t_1 \leq s_2 < t_2$,

$$f(t)\begin{cases} \leq 0, & t \in [s_1, t_1], \\ \geq 0, & t \in [s_2, t_2]. \end{cases}$$

V) *Eq.* (3.1) *is oscillatory on the intervals* $[s_k, t_k],\ k = 1, 2$.

Then Eq. (1.1) *is oscillatory.*

Proof. Suppose Eq. (1.1) is not oscillatory. Then there exists its a solution $\phi(t)$ and $t_1 \geq t_0$ such that $\phi(t) \neq 0,\ t \geq t_1$. Therefore it follows from the condition III) that

$$\phi(\alpha_j(t)) \neq 0, \quad t \geq T, \quad j = \overline{1,n}, \tag{3.5}$$

for some $T > t_1$. Consider the equations

$$y' + \frac{1}{p(t)}y^2 + \frac{q(t)}{p(t)}y + \sum_{j=1}^{n} r_j(t)\exp\Big\{-\int\limits_{\alpha_j(t)}^{t} \frac{y(\tau)}{p(\tau)}d\tau\Big\} = \frac{f(t)}{\phi(T)}, \quad t \geq T, \tag{3.6}$$

$$y' + \frac{1}{p(t)}y^2 + \frac{q(t)}{p(t)}y + \sum_{j=1}^{n} r_{1,j}(t)\exp\Big\{-\int\limits_{\alpha_j(t)}^{t} \frac{y(\tau)}{p(\tau)}d\tau\Big\} = 0, \quad t \geq T. \tag{3.7}$$

By Lemma 2.1 it follows from (3.5), that Eq. (3.6) has a solution on $[T, \infty)$. Suppose $\phi(t) > 0$ (< 0), $t \geq t_1$. Then $\dfrac{f(t)}{\phi(T)} \leq 0,\ t \in [s_1, t_1]$ $(t \in [s_2, t_2])$. By Lemma 2.3 it follows from here and the conditions I) and III) that Eq. (3.7) has a solution $y_1(t)$ on $[s_1, t_1]$ (on $[s_2, t_2]$). By (2.6) it follows from here that $\phi_1(t) \equiv \exp\Big\{\displaystyle\int_T^t \frac{y_1(\tau)}{p(\tau)}\Big\}$, $t \leq t_1$ $(t \leq t_2)$, is a nonoscillating on $[s_1, t_1]$ (on $[s_2, t_2]$) solution of Eq. (3.1). Hence, Eq. (3.1) is not oscillatory on $[s_1, t_1]$ (on $[s_2, t_2]$), which contradicts the condition V). The obtained contradiction completes the proof of the theorem.

Consider the equations

$$(p(t)\phi'(t))' + q(t)\phi'(t) + \sum_{j=1}^{n} r_j^+(t)\phi(\alpha_j(t)) = 0, \quad t \geq t_0, \tag{3.8}$$

$$(p(t)\phi'(t))' + q(t)\phi'(t) + \sum_{j=1}^{n} r_j^+(t)\phi(\alpha_j(t)) = f(t), \quad t \geq t_0. \tag{3.9}$$

The theorem is proved.

From Theorem 3.2 we obtain immediately

Corollary 3.2. *Let the conditions* III) *and* IV) *of Theorem* 3.2 *be satisfied. If Eq.* (3.8) *is oscillatory on the intervals* $[s_i, t_i]$, $i = 1, 2$, *then Eq.* (3.9) *is oscillatory.*

In the case $n = 1,\ \alpha_1(t) = t$ and continuous $r_{1,1}(t) = r_1(t),\ t \geq t_0$, Theorem 3.2 is the same as Corollary 3.1 from [6]. It was shown in [6] that Corollary 3.1 of [6] is a generalization of Theorem 1.1. Therefore Theorem 3.2 is another generalization of Theorem 1.1.

Example 3.2. Let $c_k(t),\ h_k(t),\ k = \overline{1,n}$, be real-valued locally integrable functions on $[0, \infty)$. Consider the equation

$$\phi''(t) + \sum_{k=1}^{n} c_k(t)\phi(t - h_k(t)) = \sin(t/3), \quad t \geq 0. \tag{3.10}$$

Assume $c_k(t) \equiv 0,\ t \in [3\pi l, 3\pi l + 1],\ k = \overline{1,n},\ c_k(t) \geq 0,\ \sum\limits_{k=1}^{n} c_k(t) \geq 2,\ t \in [3\pi l + 1, 3\pi(l+1)],\ l = 0, 1, 2, \dots,\ 0 \leq h_k(t) \leq 1/2,\ t \geq 0,\ k = \overline{1,n}$. We set: $t_{1,l} \equiv 3\pi l + 1/2,\ t_{2,l} \equiv (3l+2)\pi + 1/2,\ t_{3,l} \equiv (3l+2)\pi + 1,\ t_{4.l} \equiv 3\pi(l+1),\ l = 0, 1, \dots$. Then

$$T_{1,l} \equiv \inf_{\substack{t\in[t_{1,l},t_{4,l}]\\ k\in\Omega}} (t - h_k(t)) \geq 3\pi l; \quad T_{2,l} \equiv \sup_{\substack{t\in[t_{1,l},t_{4,l}]\\ k\in\Omega}} (t - h_k(t)) \leq 3\pi(l+1), \quad l = 0, 1, \dots. \tag{3.11}$$

It is not difficult to verify that for $t_s = t_{s,l}$, $s = \overline{1,4}$, the conditions of Theorem 2.2 for the homogeneous equation

$$\phi''(t) + \sum_{k=1}^{n} c_k(t)\phi(t - h_k(t)) = 0, \quad t \geq 0, \tag{3.12}$$

are satisfied for every $l = 0, 1, \ldots$. Then by Theorem 2.2 it follows from (3.11) that the last equation is oscillatory on each of intervals $[3\pi l, 3\pi(l+1)]$, $l = 0, 1, 2, \ldots$. It follows from here that the conditions of Theorem 3.2 for equations (3.10) and (3.12) are satisfied. Therefore, Eq. (3.10) is oscillatory. □

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