Trudy Instituta Matematiki i Mekhaniki UrO RAN

Vol. 31 No. 3 2025

OBSERVATION CONTROL PROBLEM FOR DIFFERENTIAL EQUATIONS[1]

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We consider a controlled linear differential equation. The controller must transfer the initial state x_0 of the equation to a given final state x_T. This process is followed by the observer, who tries to determine x_T but does not know the state vector of the equation and obtains information via the vector $y(t)$ connected with $x(t)$. With the aid of the signal $y(t)$, the observer can determine an information set containing x_T. In the case of special constraints for controls (or disturbances from the point of view of the observer), the information set becomes the ellipsoid, the parameters of which are described by the system of differential equations. In the game, the controller, who is the main player, endeavors to accomplish its task and maximize the information set simultaneously. An example is considered.

Keywords: guaranteed estimation, information set, reachable set, observation control.

Б. И. Ананьев. Задача управления наблюдением для дифференциальных уравнений.

Рассматривается управляемое линейное дифференциальное уравнение. Управляющее лицо должно переместить начальное состояние x_0 уравнения в фиксированное конечное состояние x_T. Этот процесс контролируется наблюдателем, который пытается определить x_T, но не знает фазовый вектор уравнения и получает информацию от вектора $y(t)$, связанного с $x(t)$. С помощью сигнала $y(t)$ наблюдатель может определить информационное множество, содержащее x_T. В случае специальных ограничений на управления (или возмущений с точки зрения наблюдателя) информационное множество становится эллипсоидом, параметры которого описываются системой дифференциальных уравнений. Управляющее лицо, которое является основным, пытается выполнить свою задачу и одновременно максимизировать размер информационного множества. Рассмотрен пример.

Ключевые слова: гарантированное оценивание, информационное множество, множество достижимости, управление наблюдением.

MSC: 93B99

DOI: 10.21538/0134-4889-2025-31-3-fon-02

1. Introduction and Preliminaries

In this paper, we use an approach to guaranteed estimation from [1]. In many estimation problems from mechanics, economics, biology, and financial mathematics, there are both stochastic disturbances in the system and the observation's channel and uncertain ones with unknown statistics. In particular, the stochastic part may be absent in special case of set-membership description of uncertainty, [2; 3]. In this paper, a controller uses uncertain disturbances in the system as control actions to produce worst signals for an observer, or, along with this task, to achieve his own aims unknown for the observer. On the other hand, the observer applies a minimax state estimation algorithm and does not know the aims of the controller. Such problems arise, for example, in aviation, when the plane must do some work to go unnoticed. Besides, there are other examples in economics, financial mathematics, and biology. Problems of optimization of observation's process were considered in various formulations in [4–8].

Here we continue the works [9; 10], but for more general form of the system and constraints. Namely, suppose that the dynamics of our partly observed system is described by equations:

$$\dot{x}(t) = A(t)x(t) + b(t)v(t), \quad y(t) = G(t)x(t) + cv(t), \quad 0 \le t \le T, \tag{1.1}$$

[1]The work was performed as part of research conducted in the Ural Mathematical Center with the financial support of the Ministry of Science and Higher Education of the Russian Federation (Agreement number 075-02-2025-1549).

where $x(t) \in \mathbb{R}^n$ is a state vector, $y(t) \in \mathbb{R}^m$ is an output, $v(t) \in \mathbb{R}^l$ is an uncertain disturbance; $A(\cdot)$, $G(\cdot)$, $b(\cdot)$ are continuous matrices. The observer does not know initial state x_0 and believes that uncertain functions $v(\cdot) \in L_2^l[0,T]$ in (1.1) are restricted by the integral constraints

$$J_T(v) = \int_0^T \left(|v(t)|^2 + 2s'(t)x(t) - 2r'(t)v(t)\right) dt \le 1, \tag{1.2}$$

where $|\cdot|$ is the Euclidean norm; elements of vector-functions $s(\cdot)$, $r(\cdot)$ belongs to the space $L_2[0,T]$; the symbol $'$ means the transposition. Hereinafter, by $|x|_P^2$ is denoted a quadratic form $x'Px$, where the matrix P is such that $P' = P \ge 0$. The matrix c being constant has a full rank, i.e. $\mathrm{rank}(c) = m \wedge l$, where $m \wedge l = \min\{m, l\}$.

One can see that the variables x and y from equations (1.1) and (1.2) are bound with each other by means of the function $v(t)$. Let us present the system in the equivalent form. Consider the pseudoinverse c^+ matrix to c, [12]. It is known that c^+c is the orthogonal projection onto subspace $\mathrm{im}\, c' = \{v : v = c'y,\ y \in \mathbb{R}^m\}$. Introduce the matrix $C_1 = I_l - c^+c$ that is the orthogonal projection onto null-subspace $\ker c = \{v : cv = 0\}$. Then $v(t) = c^+cv(t) + C_1v(t)$ and $cv(t) = y(t) - Gx(t)$. If we introduce a notation

$$\mathbf{b}(\cdot) = b(\cdot)c^+, \quad \mathbf{A}(\cdot) = A(\cdot) - \mathbf{b}(\cdot)G(\cdot),$$

and substitute the orthogonal expansion of $v(t)$ into (1.1), this equation is converted to the following one

$$\dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{b}(t)y(t) + b(t)C_1v(t). \tag{1.3}$$

Constraints may be rewritten as

$$\begin{aligned} J_T(v, y) = \int_0^T \Big(& |y(t) - G(t)x(t)|_C^2 + |v(t)|_{C_1}^2 + 2s'(t)x(t) \\ & - 2r'(t)\left(c^+(y(t) - G(t)x(t)) + C_1v(t)\right)\Big) dt \le 1, \quad C = (c^+)'c^+. \end{aligned} \tag{1.4}$$

In the case $\mathrm{rank}(c) = m$, we have $c^+ = c'(cc')^{-1}$ and $C = (cc')^{-1}$. In other case $\mathrm{rank}(c) = l$, we obtain $c^+ = (c'c)^{-1}c'$ and $C_1 = O_l$, i.e. zero matrix. In the last case we deal with the unique uncertain element x_0. This last case has no interest for controller because he knows x_0 and cannot change the signal. Therefore, suppose that $\mathrm{rank}(c) = m < l$. But then we can pass to lower dimension of disturbances according to the following remark.

Remark 1. As $\ker c + \mathrm{im}\, c' = l$, $\mathrm{im}\, C_1 = \ker c$, and $\mathrm{im}\, c' = m$, then $\mathrm{rank}\, C_1 = l - m$. Using expansion $C_1 = T\widetilde{C}_1T'$, where T is an orthogonal matrix, $TT' = T'T = I_l$, and $\widetilde{C}_1$ is a diagonal matrix with 0 or 1 on the diagonal, we can eliminate m zero columns from $\widetilde{C}_1$ and obtain a matrix $\widetilde{D}_1$. Then $\widetilde{C}_1 = \widetilde{D}_1\widetilde{D}_1'$ and $C_1 = D_1D_1'$, where $D_1 = T\widetilde{D}_1$. If we define vector-function $w(t) = D_1'v(t) \in \mathbb{R}^{l-m}$, then we obtain the equality

$$C_1v(t) = D_1w(t), \quad D_1 \in \mathbb{R}^{l\times(l-m)}, \quad \mathrm{rank}\, D_1 = l - m, \quad D_1'D_1 = I_{l-m}.$$

Therefore, we can use $D_1w(t)$ in (1.3), (1.4) instead $C_1v(t)$.

2. The Problem for the Observer

We introduce some definitions.

Definition 1. Let the signal $y(t)$ be generated by (1.1), (1.2) with the help of unknown pair $(x_0^*, v^*(\cdot))$ satisfying constraints. A pair $(x_0, v(\cdot))$ is called *compatible* with the measured signal $y(t)$ on $[0,T]$ if the solution $x(t)$ of equation (1.3) and the function $v(t)$ satisfy relations (1.4).

Definition 2. The set $\mathbb{X}_T(y)$ is called the *information set* (shortly IS) if it consists of all vectors $x(T)$ for each of which there exists a generating compatible pair $(x_0, v(\cdot))$ such that the corresponding trajectory $x(t)$ ends at $x(T)$.

The observer's problem is to find $\mathbb{X}_T(y)$ and to give an analytical description of this set. As it was proved by dynamic programming methods in [11], the IS $\mathbb{X}_T(y)$ under restrictions (1.4) represents the ellipsoid given by inequality

$$\mathbb{X}_T(y) = \left\{x \in \mathbb{R}^n\colon x'P(T)x - 2x'd(T) + e(T) \le 1\right\}, \tag{2.1}$$

where the parameters can be found from differential equations

$$\begin{aligned} \dot{P}(t) &= -P(t)\mathbf{A}(t) - \mathbf{A}'(t)P(t) + G'(t)CG(t) - P(t)b(t)C_1b'(t)P(t),\\ \dot{d}(t) &= P(t)\mathbf{b}(t)y(t) - \mathbf{A}'(t)d(t) + G'(t)C(y(t) - cr(t)) + P(t)b(t)C_1(r(t)\\ &\quad - b'(t)d(t)) - s(t), \quad P(0) = 0, \quad d(0) = 0, \quad e(0) = 0,\\ \dot{e}(t) &= 2y'(t)\mathbf{b}'(t)d(t) + |y(t)|^2_C - 2r'(t)c'Cy(t) - |r(t) - b'(t)d(t)|^2_{C_1}. \end{aligned} \tag{2.2}$$

The value in the right side of inequality in (2.1) equals $\min_{w(\cdot)} J_T(w, y)$ under condition $x(T) = x$. On another we can rewrite inequality in (2.1) as

$$\begin{aligned} \mathbb{X}_T(y) &= \left\{x \in \mathbb{R}^n\colon\ |x - \hat{x}(T)|^2_{P(T)} + h(T) \le 1\right\},\\ \hat{x}(T) &= P^+(T)d(T), \quad h(T) = e(T) - d'(T)P^+(T)d(T), \end{aligned} \tag{2.3}$$

where P^+ is the pseudoinverse matrix. If system (1.3) is completely observed, that is

$$\int_0^t \mathbf{X}'(u,t)G'(u)G(u)\mathbf{X}(u,t)du > 0 \quad \forall t \in (0, T], \tag{2.4}$$

then $P(t) > 0$, $t > 0$. Here $\partial \mathbf{X}(u,t)/\partial u = \mathbf{A}(u)\mathbf{X}(u,t)$ and $\mathbf{X}(u,t)$ is a fundamental matrix.

Under conditions (2.4) parameters $\hat{x}$, h satisfy the differential equations

$$\begin{aligned} \dot{\hat{x}}(t) &= A(t)\hat{x}(t) + \left(b(t)c' + P^{-1}(t)G'(t)\right)C(y(t) - G(t)\hat{x}(t) - cr(t)) - P^{-1}(t)s(t) + b(t)r(t),\\ \dot{h}(t) &= \left|y(t) - G(t)\hat{x}(t) - cr(t)\right|^2_C - |r(t)|^2 + 2s'(t)\hat{x}(t). \end{aligned} \tag{2.5}$$

There is a problem with initial states $\hat{x}(0)$, $h(0)$ for these equations. Therefore, we can use other functional approach. Introduce linear integral operators

$$Y_t(y) = \int_t^T \mathbf{X}(t,u)\mathbf{b}(u)y(u)du, \quad W_t(w) = \int_t^T \mathbf{X}(t,u)b(u)D_1w(u)du,$$

and obtain the functional in (1.4) in the form

$$\begin{aligned} J_T(w,y) = \int_0^T \Big(&|y(t) - G(t)\left(\mathbf{X}(t,T)x - Y_t(y) - W_t(w)\right)|^2_C\\ &+ |w(t)|^2 + 2s'(t)\left(\mathbf{X}(t,T)x - Y_t(y) - W_t(w)\right)\\ &- 2r'(t)\left(c'C\left(y(t) - G(t)\left(\mathbf{X}(t,T)x - Y_t(y) - W_t(w)\right)\right) + D_1w(t)\right)\Big)dt. \end{aligned}$$

If we define the linear Volterra type (see [13]), self-adjoint, and coercive operator $\mathbf{K}: L_2^{l-m}[0,T] \to L_2^{l-m}[0,T]$:

$$\mathbf{K}_t(w) = D_1'b'(t)\int_0^t \mathbf{X}'(\alpha,t)G'(\alpha)CG(\alpha)W_\alpha(w)d\alpha + w(t),$$

then

$$J_T(w,y) = S - 2\int_0^T w'(t)F_t dt + \int_0^T w'(t)\mathbf{K}_t(w)dt,$$

$$S = \int_0^T \Big(|y(t) - G(t)\left(\mathbf{X}(t,T)x - Y_t(y)\right)|_C^2 + 2s'(t)\left(\mathbf{X}(t,T)x - Y_t(y)\right) \\ - 2r'(t)c'C\left(y(t) - G(t)\left(\mathbf{X}(t,T)x - Y_t(y)\right)\right)\Big)dt,$$

$$F_t = D_1'b'(t)\int_0^t \mathbf{X}'(u,t)(C(y(u) - G(u)(\mathbf{X}(u,T)x - Y_u(y))) + s(u) + G'(u)Ccr(u))du + D_1'r(t).$$

Therefore,

$$\mathbb{X}_T(y) = \left\{ x \in \mathbb{R}^n : S - \int_0^T F_t'\mathbf{K}_t^{-1}(F_\bullet)dt \le 1 \right\}. \tag{2.6}$$

Now, we can compare formulas (2.6) with (2.2) and to express parameters $P(T)$, $d(T)$, $e(T)$ through the operator $\mathbf{K}$. To do this, we stress the dependence of S, F_t on y, x, s, r, and write $S = S(y,x,s,r)$, $F_t = F_t(y,x,s,r)$.

Lemma 1. *Using notation, we have*

$$x'P(T)x = S(0,x,0,0) - \int_0^T F_t'(0,x,0,0)\mathbf{K}_t^{-1}(F_\bullet(0,x,0,0))dt,$$

$$x'd(T) = x'\int_0^T \mathbf{X}'(t,T)\left(G'(t)(C(y(t) - Y_t(y) - cr(t)) - s(t)\right)dt \\ + \int_0^T F_t'(0,x,0,0)\mathbf{K}_t^{-1}(F_\bullet(y,0,s,r))dt,$$

$$e(T) = S(y,0,s,r) - \int_0^T F_t'(y,0,s,r)\mathbf{K}_t^{-1}(F_\bullet(y,0,s,r))dt.$$

□

Anyway, the observer can build IS $\mathbb{X}_T(y)$.

3. The Problem for the Controller

Assumption 1. Suppose that there exists a function $v_*(\cdot)$ generating along with the initial state x_0 the signal $y(\cdot)$, satisfying the inequality $J_T(w_*,y) < 1$, and such that

$$x_0 = \mathbf{X}(0,T)x_T - Y_T(y) - W_T(w_*), \\ D_1w_*(t) = v_*(t) - c'C(y(t) - G(t)x(t)), \quad w_*(t) = D_1'v_*(t), \quad \text{for given vectors } x_0,\ x_T. \tag{3.1}$$

Using Assumption 1 we can claim: there is a small $\alpha > 0$ such that $\alpha|x_0|^2 + J_T(w_*, y) < 1$. Consider IS $\mathbb{X}_T^\alpha(y)$ with constraints $\alpha|x_0|^2 + J_T(w, y) \le 1$. This set is described by equations (2.1)–(2.3) and (2.5), where initial values $P(0) = \alpha I_n$, $\hat{x}(0) = 0$, $h(0) = 0$. We have the inclusion $\mathbb{X}_T^\alpha(y) \subset \mathbb{X}_T(y)$ and limiting equality $\lim_{\alpha\to 0} \mathbb{X}_T^\alpha(y) = \mathbb{X}_T(y)$. Now, we fix α and try to enlarge IS $\mathbb{X}_T^\alpha(y)$ by minimizing $h(T)$. Indeed, the volume of $\mathbb{X}_T^\alpha(y)$ depends only on $h(T)$.

Remark 2. Note that one special case of systems (1.1) is described by equalities

$$b(\cdot) = [B(\cdot)\ O_{n\times m}], \quad c = [O_{m\times r}\ I_m], \quad B(t) \in \mathbb{R}^{n\times r}, \quad l = r + m, \quad r(t) = [r_1(t); r_2(t)]. \tag{3.2}$$

Then $C = I_m$, $b(\cdot)c' \equiv 0$, $b(\cdot)D_1 = B(\cdot)$, and $b(\cdot)C_1 \equiv b(\cdot)$. In (3.2) and further we use designations from MATLAB where $[A; B]$ means the vertical concatenation of matrices A, B and $[A\ B]$ means the horizontal concatenation.

First, we consider a simplified problem for the controller under Remark 2 and equalities (3.2). Let us temporarily fix the function $[I_r\ O_{r\times m}]v_*(\cdot) = \bar{v}(\cdot)$ in (3.1). It means that we fix the corresponding trajectory $\bar{x}(t)$ and

$$\begin{gathered} X(0,T)x_T - x_0 = \int_0^T X(0,t)B(t)\bar{v}(t)dt, \quad y(t) = G(t)\bar{x}(t) + w(t) + r_2(t), \\ \int_0^T |w(t)|^2 dt \le 1 - \int_0^T \left(|\bar{v}(t)|^2 - |r_2(t)|^2 + 2s'(t)\bar{x}(t) - 2r_1(t)\bar{v}(t)\right) dt - \alpha|x_0|^2 = \bar{\delta}. \end{gathered} \tag{3.3}$$

Here $\partial X(t,u)/\partial t = A(t)X(t,u)$. Due to Assumption 1 the right side $\bar{\delta}$ of inequality in (3.3) is grater then zero, and we can use function $w(t)$ in order to enlarge IS.

Introduce a function $f(t) = y(t) - G(t)\hat{x}(t) - r_2(t) \in L_2^m[0,T]$. Any signal can be generated by equations

$$\begin{gathered} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + P^{-1}(t)\left(G'(t)f(t) - s(t)\right) + B(t)r_1(t), \\ \hat{x}(0) = 0, \quad h(0) = 0, \\ \dot{h}(t) = |f(t)|^2 - |r(t)|^2 + 2s'(t)\hat{x}(t), \quad h(T) \le 1, \end{gathered}$$

with the equality $y(t) = f(t) + G(t)\hat{x}(t) + r_2(t)$. So, $f(t) = G(t)z(t) + w(t)$, $z(t) = \bar{x}(t) - \hat{x}(t)$. Let $\mathbf{z}(t) = [z(t); \hat{x}(t)]$. The vector $\mathbf{z}$ satisfy the following differential equation

$$\dot{\mathbf{z}} = \underbrace{\begin{bmatrix} A - P^{-1}G'G & O_n \\ P^{-1}G'G & A \end{bmatrix}}_{\mathbb{A}(t)} \mathbf{z} + \underbrace{\begin{bmatrix} -P^{-1}G' \\ P^{-1}G' \end{bmatrix}}_{\mathbb{B}(t)} w + \begin{bmatrix} P^{-1}s + B(\bar{v} - r_1) \\ -P^{-1}s + Br_1 \end{bmatrix}. \tag{3.4}$$

From (3.4) we obtain the solution

$$\mathbf{z}(t) = \mathcal{Z}_t(w) + g(t),$$

where $\mathcal{Z}_t$ is a linear operator and $g(t)$ depends on x_0, s, r_1, $\bar{v}$.

Thus, we come to the following minimization problem:

$$\begin{gathered} h(T) = \int_0^T \left(|G(t)[I_n\ O_n]\mathbf{z}(t) + w(t)|^2 - |r(t)|^2 + 2s'(t)[O_n\ I_n]\mathbf{z}(t)\right) dt \to \min_{w(\cdot)}, \\ \text{under constraints} \quad \int_0^T |w(t)|^2 dt \le \bar{\delta}. \end{gathered} \tag{3.5}$$

This is a quadratic problem with linear and inequality constraints. Composed elements under integral depending on w form the linear operator

$$\mathcal{F}_t(w) = G(t)[I_n\ O_n]\mathcal{Z}_t(w) + w(t).$$

Then the minimum of functional $h(T)$ is reached at the function

$$w_0(t) = -\left(\mathcal{F}_t^*(\mathcal{F}_\cdot)\right)^+ \left(\mathcal{F}_\bullet^*(G(\cdot)[I_n\ O_n]g(\cdot))\right) + \mathcal{Z}_\bullet^*([O_n; I_n]s(\cdot)). \tag{3.6}$$

Using Kuhn–Tucker theorem we get the conclusion.

Theorem 1. *Let Assumption 1 and equalities (3.2) be valid. If $w_0(t)$ from (3.6) satisfies integral inequality in (3.5) then this function solves minimization problem in (3.5). Otherwise,*

$$w_0(t) = -\left(\mathcal{F}_t^*(\mathcal{F}_\cdot) + k\, id\right)^{-1} \left(\mathcal{F}_\bullet^*(G(\cdot)[I_n\ O_n]g(\cdot))\right) + \mathcal{Z}_\bullet^*([O_n; I_n]s(\cdot)),$$

where the Lagrange multiplier $k > 0$ may be found from the equation $\int_0^T |w_0(t)|^2 dt = \bar{\delta}$.

In solution above we take into account the fact that IS $\mathbb{X}_T^\alpha(y) \neq \varnothing$ and for given $\bar{v}(\cdot)$ there is a signal $y(\cdot)$ such that $\alpha|x_0|^2 + J_T([\bar{v}; w], y) \leq 1$ with equality (3.4). We only make a choice of the best function $w(\cdot)$ under given $\bar{v}(\cdot)$.

Generally, in the solution above we have uncertainty in the choice of function $\bar{v}(\cdot)$. In the general case (1.1), (1.2), consider the equality's constraints for functions $v(\cdot)$:

$$X(0,T)x_T - x_0 = \int_0^T X(0,t)b(t)v(t)dt. \tag{3.7}$$

Let $z(t) = x(t) - \hat{x}(t)$ and $\mathbf{z}(t) = [z(t); \hat{x}(t)]$ as above. The vector $\mathbf{z}$ satisfy the following differential equation

$$\begin{aligned}\dot{\mathbf{z}} &= \underbrace{\begin{bmatrix} A - (bc' + P^{-1}G')CG & O_n \\ (bc' + P^{-1}G')CG & A \end{bmatrix}}_{\mathbb{A}(t)} \mathbf{z} + \underbrace{\begin{bmatrix} b - (bc' + P^{-1}G')Cc \\ (bc' + P^{-1}G')Cc \end{bmatrix}}_{\mathbb{B}(t)} v \\ &+ \begin{bmatrix} (bc' + P^{-1}G')Cc - b \\ b - (bc' + P^{-1}G')Cc \end{bmatrix} r + \begin{bmatrix} P^{-1} \\ -P^{-1} \end{bmatrix} s.\end{aligned} \tag{3.8}$$

The solution of equation (3.8) can be written as $\mathbf{z}(t) = \mathcal{Z}_t(v) + \beta(t)$, where $\mathcal{Z}_t$ is a linear operator and $\beta(t)$ depends on x_0, $r\, s$. Introduce the linear operator $\mathcal{Y}_t(v) = G(t)[I_n\ O_n]\mathcal{Z}_t(v) + cv(t)$. Then we obtain the problem of minimization:

$$\int_0^T \left(|\mathcal{Y}_t(v) + G(t)[I_n\ O_n]\beta(t) - cr(t)|_C^2 - |r(t)|^2 + 2s'(t)\hat{x}(t)\right) dt \to \min_{v(\cdot)}, \tag{3.9}$$
$$\text{under constraints (3.7) and} \quad \alpha|x_0|^2 + J_T(v, y) \leq 1.$$

Problem (3.9) may be solved in the same way as simplified one (3.5), but we prefer an approximate solution with the help of discrete systems.

The Criterion in the Form of Difference. In the solution above, it can be the relation $x_T \approx \hat{x}(T)$. It may be bad for the controller because the observer always takes the center $\hat{x}(T)$ of IS $\mathbb{X}_T(y)$ as the real state of the system (so-called "the aiming point"). Therefore, consider the following minimization problem

$$I_T(v) = \int_0^T \left(|G(t)z(t) + c(v(t) - r(t))|_C^2 - r(t) + 2s'(t)\hat{x}(t)\right) dt - |z(T)| \to \min_{v(\cdot)}, \tag{3.10}$$
$$\text{under constraints (3.7).}$$

This functional is not convex and not smooth, but we can represent it as

$$I_T(v) = \min_{|l|\le 1} \min_{v(\cdot)} (h(T) - l'z(T)).$$

In this form, in the convex minimization over v under given l it is necessary to consider additional inequality constraint of type (1.2), that is $\alpha|x_0|^2 + \int_0^T \big(|v(t)|^2 + 2s'(t)x(t) - 2r'(t)v(t)\big)\, dt \le 1$. Problem (3.10) with additional inequality constraint is solved numerically.

4. Finite Dimensional Approximation

For simplicity, we believe that $s(t) \equiv 0$, $r(t) \equiv 0$ in (1.2), (1.4), $y(\cdot) \in L_\infty^m$ and the function $b(t)$ is Lipschitzian in t. In particular, it is so if $b(t) = \text{const}$. Let us divide the segment $0 \le t \le T$ into N parts $[t_{k-1}, t_k]$ which have the same length $\delta = t_k - t_{k-1} = 1/N$, $k \in 1:N$, $t_0 = 0$, $t_N = T$. For approximation, we use piecewise constant functions, $v(t) = v_k$ saving the constant value on half-intervals $(t_{k-1}, t_k]$ and on the first segment of $[t_0, t_1]$, that is continuous from the left. Introduce the notation:

$$\begin{aligned} A_k &= X(t_k, t_{k-1}), \quad b_k = \int_{t_{k-1}}^{t_k} X(t_k, t)b(t)dt, \\ G_k &= \int_{t_{k-1}}^{t_k} G(t)X(t, t_{k-1})dt, \quad c_k = \int_{t_{k-1}}^{t_k} G(t) \int_{t_{k-1}}^{t} X(t,u)b(u)du + c\delta. \end{aligned} \tag{4.1}$$

Using (4.1), we come to the multistage partly observed system

$$x_k = A_k x_{k-1} + b_k v_k, \quad y_k = G_k x_{k-1} + c_k v_k, \quad k \in 1:N, \tag{4.2}$$

and constraints

$$J_N(v) = \delta \sum_{k\in 1:N} |v_k|^2 \le 1.$$

Remark 3. It is known that the set of piecewise constant functions is dense in the Hilbert space $L_2^l[0,T]$. Therefore, taking in attention Assumption 1, we can assert that there exists a set of vectors v_k^*, $k \in 1:N$, such that

$$x_N^* = A_{1:N}x_0 + \sum_{k\in 1:N} A_{k+1:N}b_k v_k^*, \quad J_N(v^*) < 1, \quad x_N^* \approx x_T, \quad A_{k+1:N} = A_N \cdots A_{k+1}, \quad A_{N+1:N} = I_n.$$

Using Remark 3 we can claim: there is a small $\alpha > 0$ such that $\alpha|x_0|^2 + J_N(v^*) < 1$. Consider IS $\mathbb{X}_N^\alpha(y)$ with constraints $\alpha|x_0|^2 + J_N(v) \le 1$ for system (4.2).

4.1. Information Set for the Multistage System

For convenience, we make replacement of variables:

$$\begin{gathered} \overline{x}_k = \sqrt{\alpha}x_k, \quad \overline{v}_k = \sqrt{\delta}v_k, \\ \overline{x}_k = A_k\overline{x}_{k-1} + \overline{b}_k\overline{v}_k, \quad y_k = \overline{G}_k\overline{x}_{k-1} + \overline{c}_k\overline{v}_k, \quad k \in 1:N, \\ \overline{b}_k = b_k\sqrt{\alpha/\delta}, \quad \overline{G}_k = G_k/\sqrt{\alpha}, \quad \overline{c}_k = c_k/\sqrt{\delta}, \\ J_N(\overline{v}) = |\overline{x}_0|^2 + \sum_{k\in 1:N} |\overline{v}_k|^2 \le 1. \end{gathered} \tag{4.3}$$

To find the IS $\overline{\mathbb{X}}_1^\alpha(y)$ for system (4.3) we define the function

$$V_1(y,\overline{x}_1) = \inf\left\{|\overline{x}_0|^2 + |\overline{v}_1|^2 : \overline{x}_1 = \mathbf{A}_1[\overline{x}_0;\overline{v}_1], \quad y_1 = \mathbf{B}_1[\overline{x}_0;\overline{v}_1]\right\}, \\ \mathbf{A}_1 = [A_1\ \overline{b}_1], \quad \mathbf{B}_1 = [\overline{G}_1\ \overline{c}_1], \tag{4.4}$$

where the infimum is taken over all pairs $[\overline{x}_0;\overline{v}_1]$ with fixed y_1. At the first step, we figure out the support function (see [14]) of the IS

$$\overline{\mathbb{X}}_1^\alpha(y) = \{x : V_1(y,x) \le 1\}.$$

For this purpose, we find all solutions to the equality with y_1 in (4.3). All such solutions satisfy the inclusion

$$[\overline{x}_0;\overline{v}_1] \in Y_1 + \ker \mathbf{B}_1, \quad Y_1 = \mathbf{B}_1^+ y_1.$$

As matrix $\overline{\mathbf{B}}_1 = I_{n+l} - \mathbf{B}_1^+\mathbf{B}_1$ is the orthogonal projection on $\ker \mathbf{B}_1$, we take $n_1 \le n + l$ linear independent columns of $\overline{\mathbf{B}}_1$ and compose the matrix P_1. Then we have $[\overline{x}_0;\overline{v}_1] = Y_1 + P_1 u_1$ for some $u_1 \in \mathbb{R}^{n_1}$. This means that the infimum in (4.3) should be taken over vectors of the form $[\overline{x}_0;\overline{v}_1] = Y_1 + P_1 u$, where $u \in \mathbb{R}^{n_1}$. Computing function (4.4), we get

$$V_1(y,\overline{x}_1) = \begin{cases} |Y_1|^2 + |\overline{x}_1 - \hat{x}_1|^2_{\hat{P}_1^+}, & \text{if } \overline{x}_1 \in \operatorname{im} \mathbf{A}_1, \\ +\infty & \text{if } \overline{x}_1 \notin \operatorname{im} \mathbf{A}_1. \end{cases} \\ \hat{x}_1 = \mathbf{A}_1 Y_1, \quad \hat{P}_1 = \mathbf{A}_1 P_1 (P_1' P_1)^{-1} P_1' \mathbf{A}_1'. \tag{4.5}$$

By the way, $\det A_1 \ne 0$ and $\hat{P}_1' = \hat{P}_1 > 0$, $\operatorname{im} \mathbf{A}_1 = \mathbb{R}^n$. Therefore, the support function has the form

$$\rho_{\overline{\mathbb{X}}_1^\alpha(y)}(z) = \max_{x \in \overline{\mathbb{X}}_1^\alpha(y)} z'x = z'\hat{x}_1 + \sqrt{(1 - |Y_1|^2)\, z'\hat{P}_1 z}.$$

This means that the convex compact set $\overline{\mathbb{X}}_1^\alpha(y)$ has internal points.

On the second stage we consider the function

$$V_2(y,\overline{x}_2) = \inf\left\{V_1(y,\overline{x}_1) + |\overline{v}_2|^2 : \overline{x}_2 - A_2\hat{x}_1 = \mathbf{A}_2[r_1;\overline{v}_2],\ y_2 - \overline{G}_2\hat{x}_1 = \mathbf{B}_2[r_1;\overline{v}_2]\right\},$$
$$\mathbf{A}_2 = [A_2\hat{P}_1^{1/2}\ \overline{b}_2], \quad \mathbf{B}_2 = [\overline{G}_2\hat{P}_1^{1/2}\ \overline{c}_2],$$

similarly to (4.5). The IS is described by the relation $\overline{\mathbb{X}}_2^\alpha(y) = \{x : V_2(y,x) \le 1\}$. Here we introduced a new variable $r_1 = \hat{P}_1^{-1/2}(\overline{x}_1 - \hat{x}_1)$ which satisfy the inequality $|Y_1|^2 + |r_1|^2 + |\overline{v}_2|^2 \le 1$. We have $[r_1;\overline{v}_2] = Y_2 + P_2 u_2$ for some $u_2 \in \mathbb{R}^{n_2}$, where $Y_2 = \mathbf{B}_2^+(y_2 - \overline{G}_2\hat{x}_1)$, P_2 is composed from n_2 linear independent columns of matrix $\overline{\mathbf{B}}_2 = I_{n+l} - \mathbf{B}_2^+\mathbf{B}_2$. Computing the minimum, we get

$$V_2(y,\overline{x}_2) = |Y_1|^2 + |Y_2|^2 + |\overline{x}_2 - \hat{x}_2|^2_{\hat{P}_2^+}, \quad \hat{x}_2 = A_2\hat{x}_1 + \mathbf{A}_2 Y_2, \quad \hat{P}_2 = \mathbf{A}_2 P_2 (P_2' P_2)^{-1} P_2' \mathbf{A}_2'.$$

Continuing by induction, we obtain at the k-th stage the equations and the inequality for IS $\overline{\mathbb{X}}_k^\alpha(y)$:

$$\begin{gathered} \hat{P}_0 = I_n, \quad \hat{x}_0 = 0, \\ \mathbf{A}_k = [A_k\hat{P}_{k-1}^{1/2}\ \overline{b}_k], \quad \mathbf{B}_k = [\overline{G}_k\hat{P}_{k-1}^{1/2}\ \overline{c}_k], \quad Y_k = \mathbf{B}_k^+(y_k - \overline{G}_k\hat{x}_{k-1}), \\ \hat{x}_k = A_k\hat{x}_{k-1} + \mathbf{A}_k Y_k, \quad \hat{P}_k = \mathbf{A}_k P_k (P_k' P_k)^{-1} P_k' \mathbf{A}_k', \\ P_k \quad \text{is composed from } n_k \text{ linear independent columns of the matrix} \\ \overline{\mathbf{B}}_k = I_{n+l} - \mathbf{B}_k^+\mathbf{B}_k, \quad \hat{x}_k = A_k\hat{x}_{k-1} + \mathbf{A}_k Y_k, \\ V_k(y,\overline{x}_k) = \sum_{i \in 1:k} |Y_i|^2 + |\overline{x}_k - \hat{x}_k|^2_{\hat{P}_k^+}, \quad \overline{\mathbb{X}}_k^\alpha(y) = \{x : V_k(y,x) \le 1\}. \end{gathered} \tag{4.6}$$

It can be proved that the parameters of ellipsoid $\overline{\mathbb{X}}_k^\alpha(y)$ do not depend on the choice of matrices P_i, $i \in 1:k$. Moreover, it is proved in [11, Theorem 6] that IS $\mathbb{X}_k^\alpha(y) = \{x : V_k(y, \sqrt{\alpha}x) \le 1\}$ tends to IS $\mathbb{X}_T^\alpha$ of continuous system in Hausdorff metric if $N \to \infty$.

4.2. The Approximate Problem for the Controller

Let $z_k = \overline{x}_k - \hat{x}_k$ and $\mathbf{z}_k = [z_k; \hat{x}_k]$. The vector $\mathbf{z}_k$ satisfy the following multistage equation

$$\mathbf{z}_k = \underbrace{\begin{bmatrix} A_k - \mathbf{A}_k\mathbf{B}_k^+\overline{G}_k & O_n \\ \mathbf{A}_k\mathbf{B}_k^+\overline{G}_k & A_k \end{bmatrix}}_{\mathbb{A}_k} \mathbf{z}_{k-1} + \underbrace{\begin{bmatrix} \overline{b}_k - \mathbf{A}_k\mathbf{B}_k^+\overline{c}_k \\ \mathbf{A}_k\mathbf{B}_k^+\overline{c}_k \end{bmatrix}}_{\mathbb{B}_k} \overline{v}_k. \tag{4.7}$$

The solution of equation (4.7) can be written as $\mathbf{z}_k = \mathcal{Z}_k(\overline{v}) + \beta_k$, where $\mathcal{Z}_k$ is a linear operator and $\beta_k = \mathbb{A}_{1:k}[\overline{x}_0; 0]$, $\beta_0 = [\overline{x}_0; 0]$. Introduce the linear operator $\mathcal{Y}_k(\overline{v}) = \mathbf{B}_k^+\overline{G}_k[I_n\ O_n]\mathcal{Z}_{k-1}(\overline{v}) + \overline{c}_k\overline{v}_k$, $\mathcal{Z}_0(\overline{v}) = 0$. Then we obtain the problem of minimization:

$$\begin{gathered} \sum_{k\in 1:N} \left|\mathcal{Y}_k(\overline{v}) + \mathbf{B}_k^+\overline{G}_k\beta_{k-1}\right|^2 \to \min_{\overline{v}} \\ \text{under constraints} \quad \overline{x}_N^* - A_{1:N}\overline{x}_0 = \sum_{k\in 1:N} A_{k+1:N}\overline{b}_k\overline{v}_k, \\ \text{and} \quad J_N(\overline{v}) = |\overline{x}_0|^2 + \sum_{k\in 1:N} |\overline{v}_k|^2 \le 1. \end{gathered} \tag{4.8}$$

Similarly to (3.10) we can consider the problem

$$\begin{gathered} I_N(\overline{v}) = \sum_{k\in 1:N} \left|\mathcal{Y}_k(\overline{v}) + \mathbf{B}_k^+\overline{G}_k\beta_{k-1}\right|^2 - |z_N| \to \min_{\overline{v}_\bullet}, \\ \text{under constraints in (4.8) and} \quad |\overline{x}_0|^2 + \sum_{k\in 1:N} |\overline{v}_k|^2 \le 1. \end{gathered} \tag{4.9}$$

Let us formulate the final result on approximation.

Theorem 2. *Let a control $v_0(\cdot)$ transfer an initial point x_0 of system (1.1), (1.2) to end position x_T with maximal size of IS $\mathbb{X}_T^0(y)$. Let inequality (2.4) be fulfilled as well. Then for every $\epsilon > 0$ there exist parameters $\alpha > 0$, N, and vectors $\overline{v}_{1:N}$ such that the Hausdorff distance $h(\mathbb{X}_T^0(y), \mathbb{X}_N^\alpha(y)) < \epsilon$, $|x_N - x_T| < \epsilon$, and the minimum in problem (4.8) is less then $h_T^0 + \epsilon$.*

5. An Example

We take the system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = v(t) + w(t), \quad y(t) = x_1 + v(t), \quad \int_0^T \big(v(t)^2 + w(t)^2\big)\,dt \le 1. \tag{5.1}$$

Let $T = 3$, $v(t) \equiv 1/3$, $w(t) = \sqrt{t}/3$ for numerical calculations. Integral in (5.1) equals $T/9 + T^2/18 = 5/6 < 1$. If $x_0 = [0; 1]$, the system transfers vector x_0 to $x_T = [5.8856; 3.1547]$. Then we can take $\alpha < 1/6$ in order to provide inequality

$$\alpha|x_0|^2 + \int_0^T \big(v(t)^2 + w(t)^2\big)\,dt < 1.$$

So, Assumption 1 is valid. The observer builds IS $\mathbb{X}_T^\alpha(y)$ according to equations (2.2), (2.5):

$$\begin{gathered} \dot{\hat{x}}(t) = A\hat{x}(t) + \big(bc' + P^{-1}(t)G'\big)\,(y(t) - G\hat{x}(t)), \quad \hat{x}(0) = 0, \\ \dot{h}(t) = \left|y(t) - G\hat{x}(t)\right|^2, \quad h(0) = 0, \\ \dot{P}(t) = -P(t)\mathbf{A} - \mathbf{A}'P(t) + G'G - P(t)bC_1b'P(t), \quad P(0) = \alpha I_2. \end{gathered}$$

Here $C = 1$, $A = [0\ 1; 0\ 0]$, $c = [1\ 0]$, $b = [0\ 0; 1\ 1]$, $G = [1\ 0]$, $\mathbf{A} = A - bc'G$, $C_1 = I_2 - c'c$. We introduce a function $f(t) = y(t) - G(t)\hat{x}(t) \in L_2^m[0,T]$. Any signal can be generated by equations

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \left(bc' + P^{-1}(t)G'\right) f(t), \quad \hat{x}(0) = 0, \quad h(0) = 0,$$
$$\dot{h}(t) = |f(t)|^2, \quad h(T) \le 1,$$

with the equality $y(t) = f(t) + G(t)\hat{x}(t)$. So,

$$\hat{x}(t) = \int_0^t X(t,u)\left(bc' + P^{-1}G'\right) f(u)du = \mathcal{H}_t(f).$$

Let us temporarily fix the function $w(\cdot) = \sqrt{t}/3$. We come to the following minimization problem:

$$h(T) \to \min_{f(\cdot)}, \quad \text{under constraints} \quad \mathbf{X}(0,T)x_T - x_0$$
$$= \int_0^T \mathbf{X}(0,t)\left(\mathbf{b}(f(t) + G(t)\hat{x}(t)) + bD_1 w(t)\right) dt, \quad \mathbf{b} = bc', \quad D_1 = [0;1]. \tag{5.2}$$

We solve the problems (5.2), (3.9), and (3.10) using finite dimensional approximation. For this purpose we set $N = 15$, $\delta = T/N = 0.2$. From (4.1) we get

$$A = [1\ \delta; 0\ 1], \quad b = [\delta^2/2\ \delta^2/2; \delta\ \delta], \quad G = [\delta\ \delta^2/2], \quad c = [\delta + \delta^3/6\ \delta^3/6],$$

and a multistage system with constant coefficient:

$$x_k = Ax_{k-1} + bv_k, \quad y_k = Gx_{k-1} + cv_k, \quad k \in 1:N, \quad J_N(v) = \delta \sum_{k \in 1:N} |v_k|^2 \le 1.$$

The transition between x_0 and x_T can be fulfilled by control actions $v_k^0 = b'(A^{N-k})'\gamma/\delta$, where

$$\gamma = T_N^{-1}(x_T - A^N x_0), \quad T_N = \sum_{k=1:N} A^{N-k}bb'(A^{N-k})'/\delta,$$
$$J_N(v^0) = (x_T - A^N x_0)'T_N^{-1}(x_T - A^N x_0) = 0.8006 < 1.$$

Therefore, we can pass to constraints $\alpha|x_0|^2 + J_N(v) \le 1$, where $\alpha < 0.1667$. After the replacement (4.3) we have

$$\overline{x}_k = A\overline{x}_{k-1} + \overline{b}\overline{v}_k, \quad y_k = \overline{G}\overline{x}_{k-1} + \overline{c}\,\overline{v}_k, \quad k \in 1:N,$$
$$\overline{b} = b\sqrt{\alpha/\delta}, \quad \overline{G} = G/\sqrt{\alpha}, \quad \overline{c} = c/\sqrt{\delta},$$
$$[1ex] J_N(\overline{v}) = |\overline{x}_0|^2 + \sum_{k \in 1:N} |\overline{v}_k|^2 \le 1, \quad \mathbf{A} = [A\ \overline{b}], \quad \mathbf{B} = [\overline{G}\ \overline{c}].$$

Further, $\overline{\mathbf{B}} = I_4 - \mathbf{B}^+\mathbf{B}$ and so on according (4.6). Numerical calculations for controller's problem (4.9) give $\sum_{k \in 1:N} |Y_k|^2 = 0.3332$ with $|z_N| = 0.3124$. For controller's problem (4.8), we have $\sum_{k \in 1:N} |Y_k|^2 = 0.3265$ with $|z_N| = 0.2959$.

Conclusion

- In this work, we consider estimation and control problems for linear systems with observation.
- The controller have to move an initial state of the equation to a given final state. This process has been noticed by the observer who tries to define the final state of the system absorbing information from the measurement.

• A controller uses uncertain disturbances in the system as control actions to produce worst signals for an observer, or, along with this task, to achieve his own aims unknown for the observer. The solution of controller's problem reduces to a quadratic minimization problem with equality and inequality constraints.

• Such problems arise, for example, in aviation, when the plane must do some work to go unnoticed. Besides, there are other examples in economics, financial mathematics, and biology.

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Received April 25, 2025
Revised May 12, 2025
Accepted May 12, 2025
Published online June 19, 2025

Funding Agency: The work was performed as part of research conducted in the Ural Mathematical Center with the financial support of the Ministry of Science and Higher Education of the Russian Federation (Agreement number 075-02-2025-1549).

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Cite this article as: B. I. Ananyev. Observation control problem for differential equations. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2025, vol. 31, no. 3, pp. 36–46.