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On a Control Reconstruction Problem with Nonconvex Constraints

N. N. Subbotina[1,2,*] **and E. A. Krupennikov**[1,2,**]

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Abstract—A control reconstruction problem for dynamic deterministic affine-control systems is considered. This problem consists of constructing piecewise constant approximations of an unknown control generating an observed trajectory from discrete inaccurate measurements of this trajectory. It is assumed that the controls are constrained by known nonconvex geometric constraints. In this case, sliding modes may appear. To describe the impact of sliding modes on the dynamics of the system, the theory of generalized controls is used. The notion of normal control is introduced. It is a control that generates an observed trajectory and is defined uniquely. The aim of reconstruction is to find piecewise constant approximations of the normal control that satisfy given nonconvex geometric constraints. The convergence of approximations is understood in the sense of weak convergence in the space L^2. A solution to the control reconstruction problem is proposed.

Keywords: inverse problems, control reconstruction, sliding modes, nonconvex constraints, weak convergence, generalized controls.

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1. INTRODUCTION

This paper is devoted to inverse problems for dynamic control systems. Namely, we consider the control reconstruction problem (in what follows, CRP), in which it is required to construct piecewise constant approximations of an unknown control that generates a trajectory based on discrete inaccurate measurements of this trajectory. Dynamic deterministic affine-control systems are considered.

There are a number of modern methods for solving the CRP. A great contribution to their development was made by Arkadii Viktorovich Kryazhimskii. The monographs [1,2], among others, are close to the topic of the present research. Let us specifically mention methods based on the approach to solving the CRP proposed by Kryazhimskii and Osipov [3]. This approach employs the extremal aiming procedure, which has roots in the works of Krasovskii's school [4]. Based on this approach, a number of numerical methods have been developed (see the review [5]).

In the present paper, we consider CRPs with nonconvex geometric constraints on the controls. Under nonconvex constraints, sliding control modes may arise [6]. In this case, which has not previously been considered for CRPs, the solution of the CRP is the normal control, which is defined

[1]Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620108 Russia
[2]Ural Federal University, Yekaterinburg, 620000 Russia
e-mail: *subb@uran.ru, **krupennikov@imm.uran.ru

uniquely. Note that the methods described in [5] can be used to construct approximations of a solution (a measurable control that generates the observed trajectory and has the smallest L^2-norm) converging to it in the L^2 norm. However, we cannot guarantee the possibility of approximating this solution in the sense of the strong topology of L^2 by measurable controls that satisfy given nonconvex constraints.

Another difficulty of the CRP is that the same observed trajectory may be generated by different controls, especially if sliding controls are allowed. We propose the notion of normal control generating the observed trajectory and defined uniquely. In a well-posed statement of the CRP, the goal is to reconstruct this normal control.

We show that the normal control can always be approximated in the sense of the weak topology of the space L^2 by piecewise constant approximating controls that satisfy given nonconvex constraints.

A solution to the formulated CRP with nonconvex constraints on the control is proposed. The weak convergence of approximations constructed within this approach is proved. An estimate of the discrepancy between the trajectories generated by the approximating controls and the observed trajectory is derived.

2. INPUT DATA OF THE CRP

2.1. Dynamics. We consider dynamic affine-control deterministic systems of the form

$$\frac{dx(t)}{dt} = G(t, x(t))u(t) + f(t, x(t)),$$
$$x(\cdot) : [0, T] \to \mathbb{R}^n, \quad u(\cdot) : [0, T] \to \mathbb{R}^m, \quad t \in [0, T], \quad T < \infty. \tag{2.1}$$

There are geometric constraints on the control values

$$u(t) \in \mathbf{U}, \tag{2.2}$$

where $\mathbf{U} \subset \mathbb{R}^m$ is a nonconvex compact set.

2.2. Measurements. A trajectory $x^*(\cdot) \colon [0, T] \to \mathbb{R}^n$ of system (2.1) generated by an unknown control is observed. Information about the trajectory has the form of a set of inaccurate discrete measurements. The absolute error of the measurements is $\delta > 0$. The measurements are received with step $h > 0$. It is assumed that $h = h(\delta)$.

The points of measurements are denoted by y_i^δ:

$$\|y_i^\delta - x^*(t_i)\| \le \delta, \quad t_i = ih, \quad T = Nh, \quad i = 0, \dots, N. \tag{2.3}$$

2.3. Assumptions. The following assumptions are introduced.

1. There exist constants $d_0 > 0$, $\delta_0 > 0$, and $h_0 > 0$ and a compact set $\Psi \subset \mathbb{R}^n$ such that, for any parameters of the measurements $\delta \in (0, \delta_0]$ and $h \in (0, h_0]$, the following condition holds:

$$\bigcup_{i=0,\dots,N} B_{d_0}[y_i^\delta] \subset \Psi, \tag{2.4}$$

where $B_{d_0}[y_i^\delta]$ is a closed ball of radius d_0 centered at the measurement point y_i^δ (2.3).

2. The matrix $G(t,x)$ and the vector $f(t,x)$ in the dynamics (2.1) are continuous in time and locally Lipschitz in the state variable for $(t,x) \in D_0 \triangleq [0,T] \times \Psi$ with Lipschitz constant $L = L(D_0) > 0$:

$$\begin{gathered} \|f(t,x_2) - f(t,x_1)\| \leq L\|x_2 - x_1\|, \\ \|G(t,x_2) - G(t,x_1)\|_2 \leq L\|x_2 - x_1\| \quad \forall (t,x_1,x_2) \in D_0. \end{gathered} \tag{2.5}$$

Here we denote by $\|\cdot\|_2$ the spectral matrix defined as $\|G\|_2 \stackrel{\text{def}}{=} \max\limits_{\|x\|=1} \|Gx\|$.

3. STATEMENT OF THE CRP

The general statement of the CRP is as follows: based on sets of inaccurate measurements (2.3), construct approximations of a control that generates the observed trajectory. In order to formulate a well-posed CRP, we should introduce the notion of normal control, i.e., a control that generates the observed trajectory and is defined uniquely; such a control will be considered a solution to the CRP. In addition, we should choose and justify the type of convergence of the approximations to the normal control.

3.1. Normal control.

Generalized controls. In the case of nonconvex geometric constraints on controls (2.2), sliding control modes [6] may arise. The impact of sliding controls on the dynamics of system (2.1) is described in terms of the theory of generalized controls [7, 8].

Generalized controls are time-measurable functions $t \longrightarrow \mu_t(du) : [0,T] \to \operatorname{rpm}(\mathbf{U})$ with values in the set of regular probability Borel measures on $\mathbf{U}$ with the topology induced by the weak star topology of $C^*(\mathbf{U})$, which is the space conjugate to the space of continuous functions.

We consider a generalized dynamics generated by generalized controls:

$$\frac{dx(t)}{dt} = \int\limits_{\mathbf{U}} G(t,x(t))u\,\mu_t(du) + f(t,x(t)). \tag{3.1}$$

Averaged controls. To each generalized control $\mu_t(du)\colon [0,T] \to \operatorname{rpm}(\mathbf{U})$, we assign the *averaged control* $v(\cdot)\colon [0,T] \to \mathbb{R}^m$:

$$t \longrightarrow v(t) = \int\limits_{\mathbf{U}} u\,\mu_t(du).$$

Averaged controls have the following properties.

1. Averaged controls are measurable functions (see [8], Subsect. IV.1.6).
2. The values of averaged controls belong to the convex hull of the set $\mathbf{U}$:

$$v(t) \in \operatorname{co}\mathbf{U} \quad \text{a.e. on } [0,T].$$

3. An averaged control may correspond to more than one generalized control. Each averaged control $v(\cdot)$ corresponds to a set M_v of generalized controls:

$$M_v \triangleq \left\{ t \to \mu_t(du)\colon \int\limits_{\mathbf{U}} u\,\mu_t(du) = v(t) \text{ for a.a. } t \in [0,T] \right\}.$$

4. An averaged control $v(\cdot)$ is equivalent in terms of its action to each generalized control from the set M_v in the sense that they generate the same trajectory under identical initial conditions.

Indeed, since system (2.1) is linear in controls, we have

$$\int\limits_U G(t, x(t))u\,\mu_t(du) = G(t, x(t)) \int\limits_U u\,\mu_t(du) = G(t, x(t))v(t).$$

5. The sets of generalized and averaged controls that generate the same trajectory $x^*(\cdot)$ of system (3.1) are convex. This property is a consequence of the linearity of system (3.1) in the controls.

Convexification of the dynamics. Because of the equivalent action, the set of generalized controls M_v is identified with the corresponding averaged control $v(\cdot)$. Then the action of sliding controls can be described by the original dynamics (2.1) with convexified control constraints rather than by the generalized dynamics (3.1). Instead of constraints (2.2), we adopt the constraints

$$u(t) \in \operatorname{co}\mathbf{U} \quad \text{a.e. on } [0, T]. \tag{3.2}$$

Remark 1. We assume that the observed trajectory can be generated by a sliding control. This trajectory can be interpreted as the trajectory of system (2.1) generated by an averaged control that satisfies the convex constraints (3.2).

Normal control. Following [2, 9], we define the *normal control* as the measurable control $u^*(\cdot)$ generating the trajectory $x^*(\cdot)$ of system (2.1), satisfying the convex constraints (3.2), and having the smallest norm in the space L^2.

It follows from Property 5 and the strict convexity of the L^2 norm that the normal control is unique.

3.2. Convergence of approximations of the normal control. Let us show that we cannot require the strong convergence in the space L^2 of approximations of the normal control that satisfy the nonconvex geometric constraints (2.2). As an example, consider the generalized dynamics

$$\frac{dx(t)}{dt} = \int\limits_{\mathbf{U}} u\,\mu_t(du), \quad x, u \in \mathbb{R}, \quad \mathbf{U} = \{1; -1\} \text{ is a two-point set.} \tag{3.3}$$

Let the observed trajectory $x^*(t) \equiv 0$ be generated by the generalized control (sliding mode)

$$\mu_t(du)\colon\ \mu_t(1) = \mu_t(-1) = 0.5 \quad \forall t \in [0, T].$$

In this case, the normal control $u^*(t)$ is identically zero. However, for any piecewise constant function $u_\delta(\cdot)$ satisfying the nonconvex constraints from (3.3), we have

$$\|u_\delta(t) - u^*(t)\|_{L^2} = \sqrt{\int\limits_0^T (u_\delta(t) - u^*(t))^2 dt} = \sqrt{T} \nrightarrow 0.$$

Thus, the example shows that it is not always possible to approximate the normal control in the sense of the strong topology of the space L^2 by piecewise constant functions that satisfy given nonconvex constraints.

The convergence of approximations of the normal control is understood in the sense of the weak topology of $L^2([0,T],\mathbb{R}^m)$.

A sequence of functions $u_i(\cdot) \in L^2([0,T],\mathbb{R}^m)$ converges to a function $v(\cdot) \in L^2([0,T],\mathbb{R}^m)$ weakly in L^2 if

$$\int_0^T \langle \varphi(t), u_i(t) - v(t)\rangle dt \overset{i\to\infty}{\longrightarrow} 0 \quad \forall \varphi(\cdot) \in L^2([0,T],\mathbb{R}^m), \tag{3.4}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product.

In what follows, we use the notation

$$\begin{gathered} \mathcal{U} \triangleq \{u(\cdot) \in L^2([0,T],\mathbb{R}^m)\colon u(t) \in \mathbf{U} \quad \text{a. e. on } [0,T]\}, \\ \operatorname{co}\mathcal{U} \triangleq \{u(\cdot) \in L^2([0,T],\mathbb{R}^m)\colon u(t) \in \operatorname{co}\mathbf{U} \quad \text{a. e. on } [0,T]\}, \\ R_U \triangleq \max_{u\in \operatorname{co}\mathbf{U}} \|u\|, \quad R_G \triangleq \max_{(t,x)\in D_0} \|G(t,x)\|_2, \quad R_f \triangleq \max_{(t,x)\in D_0} \|f(t,x)\|. \end{gathered} \tag{3.5}$$

We will use the following auxiliary statement.

Assertion 1. *For functions from the set* co$\mathcal{U}$*, weak convergence in* L^2 (3.4) *is equivalent to the convergence*

$$\int_0^T \langle \xi(t), u_i(t) - v(t)\rangle dt \overset{i\to\infty}{\longrightarrow} 0 \quad \forall \xi(\cdot) \in C([0,T],\mathbb{R}^m). \tag{3.6}$$

Proof. Indeed, all function from co$\mathcal{U}$ are bounded in total by the constant R_U, and the set of continuous functions $C([0,T],\mathbb{R}^m)$ is everywhere dense in $L^2([0,T],\mathbb{R}^m)$ (see [8], I.5.18). Then the assertion is a consequence of Theorem 2 from Subsect. IV.3.2 of [10]. □

Let us prove the following theorem.

Theorem 1. *The convex hull* co$\mathcal{U}$ *of the set* $\mathcal{U}$ (3.5) *coincides with the closure of* $\mathcal{U}$ *in the weak topology of the space* L^2.

Proof. First, the set co$\mathcal{U}$ (3.5) is closed in the weak topology of the space $L^2([0,T],\mathbb{R}^m)$. This fact follows from Lemma 1A ([11], Ch. 2, Appendix), according to which the set co$\mathcal{U}$ of all measurable functions with values in the compact convex set co $\mathbf{U}$ is compact in the weak topology of the space L^2.

Second, for any element from co$\mathcal{U}$, there exists a sequence of elements from $\mathcal{U}$ that converges to it weakly in L^2. This follows from Theorem 12.6.7 in [12].

Indeed, consider the Banach separable self-conjugate (see [10], Subsect. IV.2.3.1) space $\mathbb{R}^m$. The subset co $\mathbf{U} \subset (\mathbb{R}^m)^* = \mathbb{R}^m$ is convex, bounded, and weakly closed (because strong and weak convergences are equivalent in a finite-dimensional Euclidean space (see [10], Subsect. IV.2.3.1). By construction, co $\mathbf{U}$ is the convex hull of $\mathbf{U}$. Hence, the set $\mathbf{U}$ is total in the set co $\mathbf{U}$ [12, p. 650]. Following Subsect. 12.6.7 of [12], we denote by $PC(0,T;\mathbf{U}) \subset \mathcal{U}$ the set of piecewise constant functions on $[0,T]$ with values in $\mathbf{U}$. Denote by $R(0,T;\operatorname{co}\mathbf{U})$ the set of measurable functions with values in co $\mathbf{U}$; i.e., $R(0,T;\operatorname{co}\mathbf{U}) = \operatorname{co}\mathcal{U}$. Then, by Theorem 12.6.7 from [12], the set $PC(0,T;\mathbf{U})$ is weakly in $L^1([0,T],\mathbb{R}^m)$ sequentially dense in the set $R(0,T;\operatorname{co}\mathbf{U})$.

By the definition of $L^1([0,T],\mathbb{R}^m)$-weak convergence (see Theorem 12.2.11 from [12]),

$$\forall u(\cdot) \in \mathrm{co}\,\mathcal{U} \quad \exists \{u_k(\cdot) \in PC(0,T;\mathbf{U}) \subset \mathcal{U},\ k=1,2,\ldots\}:$$
$$\int\limits_0^T \langle u(t)-u_k(t), \eta(t)\rangle dt \overset{k\to\infty}{\longrightarrow} 0 \quad \forall \eta(\cdot) \in L^\infty([0,T],\mathbb{R}^m).$$

In particular, since $C([0,T],\mathbb{R}^m) \subset L^\infty([0,T],\mathbb{R}^m)$, we have

$$\int\limits_0^T \langle u(t)-u_k(t), \xi(t)\rangle dt \overset{k\to\infty}{\longrightarrow} 0 \quad \forall \xi(\cdot) \in C([0,T],\mathbb{R}^m).$$

Then, however, it follows from Assertion 1 that the sequence $\{u_k(\cdot)\} \subset \mathcal{U}$ converges to $u(\cdot) \in \mathrm{co}\,\mathcal{U}$ weakly in L^2. □

Remark 2. Theorem 1, in particular, implies that any function from $\mathrm{co}\,\mathcal{U}$ (including the normal control) can be approximated in the sense of the weak topology of L^2 by piecewise constant functions from $\mathcal{U}$, i.e., by measurable controls satisfying the nonconvex constraints (3.2).

3.3. Well-posed statement of the CRP. The control reconstruction problem consists in the following.

It is required to construct from measurements (2.3) of the observed trajectory $x^*(\cdot)$ obtained for parameters $\delta \in (0,\delta_0]$ and $h \in (0,h_0]$ approximating piecewise constant controls $u_\delta(\cdot): [0,T] \to \mathbb{R}^m$ satisfying the following conditions.

1. They satisfy given nonconvex geometric constraints

$$u_\delta(t) \in \mathbf{U}, \quad t \in [0,T]. \tag{3.7}$$

2. The trajectories $x_\delta(\cdot)$ generated by these controls uniformly converge to the observed trajectory:

$$\|x_\delta(\cdot) - x^*(\cdot)\|_C \overset{\delta\to 0}{\longrightarrow} 0. \tag{3.8}$$

3. These controls weakly in L^2 converge to the normal control $u^*(\cdot)$:

$$\int\limits_0^T \langle \varphi(t), u_\delta(t) - u^*(t)\rangle dt \overset{\delta\to 0}{\longrightarrow} 0 \ \ \forall \varphi(\cdot) \in L^2([0,T],\mathbb{R}^m). \tag{3.9}$$

4. SOLUTION OF THE CRP

We propose an approach to solving the CRP (3.7)–(3.9) with nonconvex geometric control constraints (2.2), in which it is assumed that there is a known solution of the CRP (3.7)–(3.9) for the case of convex constraints of the form (3.2). In other words, there is a known method for the construction of auxiliary piecewise constant approximating controls $\hat{u}_\delta(\cdot) = \hat{u}_\delta(\cdot\,;\delta)\colon [0,T] \to \mathbb{R}^m$ for the normal control $u^*(\cdot)$ satisfying the convex geometric constraints (3.2) and conditions (3.8) and (3.9) of the CRP.

We assume that these approximations are constant on the time intervals $[t_i, t_{i+1}) = [ih, (i+1)h)$ and have the form

$$\hat{u}_\delta(t) = \hat{u}_{\delta,i} \in \operatorname{co}\mathbf{U}, \quad t \in [t_i, t_{i+1}), \quad i = 0, 1, \ldots, N-1. \tag{4.1}$$

Remark 3. Several methods are known for the construction of auxiliary approximations with the required properties. In particular, we mention a number of methods (see the review [5]) based on the approach proposed by Kryazhimskii and Osipov [3].

In our works [13–15], we developed and justified another approach to the construction of auxiliary approximations $\hat{u}_\delta(\cdot)$ that satisfy the convex constraints (3.2). This approach relies on the use of constructions from auxiliary problems of the calculus of variations that involve finding stationary points of pay-off integral functionals. A feature of the approach is the use in auxiliary problems of functionals whose integrands are d.c.-functions [16], i.e., differences of two convex functions. The pay-off functionals have the form

$$I(x(\cdot), u(\cdot)) = \int_0^T \Big[-\frac{\|x(t) - y^\delta(t)\|^2}{2} + \alpha^2 \frac{\|u(t) - u^*(t)\|^2}{2} \Big] dt. \tag{4.2}$$

Here $\alpha > 0$ is a small regularizing [9] parameter, and the function $y^\delta(\cdot) : [0, T] \to \mathbb{R}^n$ is a smooth interpolation of the discrete measurements (2.3). A detailed algorithm for constructing auxiliary approximations using this method was described and justified in [13–15]. To implement this algorithm, we need the following assumptions (in addition to Assumptions 1 and 2 from Subsection 2.3).

3. The dimension of the controls m is greater than or equal to the dimension of the state variables n.
4. The functions $G(\cdot)$ and $f(\cdot)$ are locally Lipschitz on D_0 with Lipschitz constant L.
5. The rank of the matrix $G(t, x)$ is n for all $(t, x) \in D_0$.

Note that the pointwise convergence of approximations of the normal control was shown in [14]. However, for bounded functions, pointwise convergence implies weak convergence (as shown in Theorem 13.44 from [17]).

Based on the argument from [14], we obtain an estimate for the discrepancy between the trajectories $\hat{x}_\delta(\cdot)\colon [0, T] \to \mathbb{R}^n$ generated by the auxiliary approximating controls $\hat{u}_\delta(\cdot)$ and the observed trajectory $x^*(\cdot)$. The estimate contains the parameter $\alpha > 0$ from the auxiliary functionals (4.2), which is an additional small parameter of this approximation method.

Lemma 1. *Let Assumptions* 1–5 *be satisfied. Let $\hat{x}_\delta(\cdot)$ be the trajectories of system* (2.1) *generated by auxiliary approximations $\hat{u}_\delta(\cdot)$ that are constructed by the method described in* [14]. *Assume that the parameters $\delta \le \delta_0$, $h = h(\delta) \le h_0$, and $\alpha = \alpha(\delta) > 0$ tend to zero and satisfy the matching conditions*

$$\alpha \xrightarrow{\delta \to 0} 0, \quad \frac{\delta}{h} \xrightarrow{\delta \to 0} 0, \quad 0 < \frac{\alpha}{h^2} \le K_0 < \infty. \tag{4.3}$$

Then

$$\|\hat{x}_\delta(\cdot) - x^*(\cdot)\|_C \xrightarrow{\delta \to 0} 0. \tag{4.4}$$

Further, assume that the parameters h and α as chosen as follows:

$$h = \sqrt{\delta}, \quad \alpha = \delta. \tag{4.5}$$

Then there exist constants K_1 and K_2, depending on the properties of the functions $G(t,x)$ and $f(t,x)$ from the dynamics (2.1), such that

$$\|\hat{x}_\delta(t) - x^*(t)\| \le K_2 e^{K_1 T}\sqrt{\delta} + o(\sqrt{\delta}), \quad t \in [0,T].$$

Proof. Recall the proof of Theorem 2 from [14, pp. 234–239]. The last formula of that proof (see [14, p. 239]) is an estimate for the discrepancy:

$$\begin{aligned}\|\hat{x}^\delta(t) - x^*(t)\| \le \Big(\delta + 2hR_GR_U + 2TLR_U(\delta + h(K+1))\Big)e^{L(R_U+1)T} \\ + 2\frac{R_G\, r_{\hat{u}}(\delta,h,\alpha)}{L(R_U+1)}\big(e^{L(R_U+1)T} - 1\big), \quad t \in [0,T],\end{aligned} \tag{4.6}$$

where $K \triangleq \max\limits_{u\in\mathbf{U},\ (t,x)\in D_0} \|G(t,x)u + f(t,x)\|$, and the function $r_{\hat{u}}(\cdot)$ is defined in formula (21) from [14]. Introduce auxiliary constants

$$C_1 \triangleq L(R_U+1), \quad C_2 \triangleq 1 + 2TLR_U,$$

$$C_3 \triangleq 2R_GR_U + 2TLR_U(K+1), \quad C_4 \triangleq \frac{2R_G}{L(R_U+1)}.$$

Rewrite estimate (4.6) using these constants:

$$\|\hat{x}^\delta(t) - x^*(t)\| \le e^{C_1T}\big(C_2\delta + C_3h + C_4 r_{\hat{u}}(\delta,h,\alpha)\big). \tag{4.7}$$

According to formula (21) from [14],

$$r_{\hat{u}}(\delta,h,\alpha) \triangleq r_k(\delta,h) + R_{G^\top Q^{-1}}\frac{r_x(\delta,h,\alpha) + 2\delta}{h}, \tag{4.8}$$

where

$$R_{G^\top Q^{-1}} \triangleq \max_{(t,x)\in D_0} \big\|G^\top(t,x)\big[G(t,x)G^\top(t,x)\big]^{-1}\big\|_2,$$

and the functions $r_k(\cdot)$ and $r_x(\cdot)$ are defined in formulas (12) and (18) from [14]. Note that the matrix $\big[G(t,x)G^\top(t,x)\big]^{-1}$ exists according to Assumption 5.

Rewrite expression (4.8) as

$$r_{\hat{u}}(\delta,h,\alpha) = r_k(\delta,h) + C_5\frac{r_x(\delta,h,\alpha)}{h} + C_6\frac{\delta}{h}, \quad C_5 \triangleq R_{G^\top Q^{-1}}, \quad C_6 \triangleq 2R_{G^\top Q^{-1}}. \tag{4.9}$$

In view of relation (16) from [14],

$$\begin{aligned}|r_k(\delta,h)| \le (\delta + h(K+1))\left(L_{G^\top Q^{-1}}(K+R_f) + R_{G^\top Q^{-1}}L\right) = C_7\delta + C_8h, \\ C_7 \triangleq \left(L_{G^\top Q^{-1}}(K+R_f) + R_{G^\top Q^{-1}}L\right), \quad C_8 \triangleq (K+1)\left(L_{G^\top Q^{-1}}(K+R_f) + R_{G^\top Q^{-1}}L\right),\end{aligned} \tag{4.10}$$

where $L_{G^\top Q^{-1}}$ is a Lipschitz constant of the matrix function $G(\cdot)\left[G(\cdot)G^\top(\cdot)\right]^{-1}$. It was shown in [14, p. 9] that such a constant exists.

Further, it follows from formula (18) in [14] that

$$r_x(\delta,h,\alpha)=\frac{T}{h}\alpha(\lambda^*)^{0.5}n\Big(\frac{L}{\lambda_*}(2\delta+h(K+1))+12\frac{\alpha}{\lambda_*^{1.5}}\frac{(2\delta+hK)}{h^2}+48\frac{\alpha^3}{\lambda_*^2}\frac{(2\delta+hK)}{(h)^3}\Big)$$
$$+n\Big(48\frac{\alpha^2}{\lambda_*}\frac{(2\delta+hK)}{h^2}+24\frac{\alpha^3}{\lambda_*^{1.5}}\frac{(2\delta+hK)}{h^3}\Big), \tag{4.11}$$

where λ_* and λ^* denote the smallest and the largest eigenvalues of the matrix $G(t,x)G^\top(t,x)$:

$$\lambda_*\triangleq\min_{(t,x)\in D_0}\lambda_{\min}(G(t,x)G^\top(t,x)),\quad \lambda^*\triangleq\max_{(t,x)\in D_0}\lambda_{\max}(G(t,x)G^\top(t,x)).$$

It is shown in [14, p. 237] that these parameters exist and $0<\lambda_*\le\lambda^*<\infty$.

Introduce auxiliary constants

$$C_9\triangleq T(\lambda^*)^{0.5}2n\frac{L}{\lambda_*},\quad C_{10}\triangleq T(\lambda^*)^{0.5}n\frac{L}{\lambda_*}(K+1),\quad C_{11}\triangleq 24Tn\frac{(\lambda^*)^{0.5}}{\lambda_*^{1.5}},$$
$$C_{12}\triangleq 2TnK\frac{(\lambda^*)^{0.5}}{\lambda_*^{1.5}},\quad C_{13}\triangleq 96Tn\frac{(\lambda^*)^{0.5}}{\lambda_*^2},\quad C_{14}\triangleq 48TnK\frac{(\lambda^*)^{0.5}}{\lambda_*^2},\quad C_{15}\triangleq 96n\frac{1}{\lambda_*},$$
$$C_{16}\triangleq 48nK\frac{1}{\lambda_*},\quad C_{17}\triangleq 48n\frac{1}{\lambda_*^{1.5}},\quad C_{18}\triangleq 24nK\frac{1}{\lambda_*^{1.5}}.$$

Rewrite expression (4.11) using these constants:

$$r_x(\delta,h,\alpha)=C_9\frac{\delta\alpha}{h}+C_{10}\alpha+C_{11}\frac{\delta\alpha^2}{h^3}+C_{12}\frac{\alpha^2}{h^2}$$
$$+C_{13}\frac{\delta\alpha^4}{h^4}+C_{14}\frac{\alpha^4}{h^3}+C_{15}\frac{\delta\alpha^2}{h^2}+C_{16}\frac{\alpha^2}{h}+C_{17}\frac{\delta\alpha^3}{h^3}+C_{18}\frac{\alpha^3}{h^2}. \tag{4.12}$$

Thus, we have obtained an estimate for the function $r_k(\cdot)$ (4.10) and an expression for $r_x(\cdot)$ (4.12). Substitute them into the estimate for $r_{\hat u}(\cdot)$ (4.9) and then into the estimate of the discrepancy of the trajectories (4.7):

$$\|\hat x^\delta(t)-x^*(t)\|\le e^{C_1T}\bigg(\delta C_2+C_4\Big(\delta C_7+hC_8+C_6\frac{\delta}{h}+C_5\Big(C_9\frac{\delta\alpha}{h^2}+C_{10}\frac{\alpha}{h}+C_{11}\frac{\delta\alpha^2}{h^4}$$
$$+C_{12}\frac{\alpha^2}{h^3}+C_{13}\frac{\delta\alpha^4}{h^5}+C_{14}\frac{\alpha^4}{h^4}+C_{15}\frac{\delta\alpha^2}{h^3}+C_{16}\frac{\alpha^2}{h^2}+C_{17}\frac{\delta\alpha^3}{h^4}+C_{18}\frac{\alpha^3}{h^3}\Big)\Big)\bigg).$$

It is easy to verify that, in this case, the assertion (4.4) of the lemma is valid if the matching conditions (4.3) are satisfied.

Taking the parameters h and α according to (4.5), we get

$$\|x^\delta(t)-x^*(t)\|\le e^{C_1T}\Big((C_4C_8+C_4C_5C_{10}+C_6+C_{12})\sqrt{\delta}+(C_2+C_4C_7+C_4C_5C_9$$
$$+C_4C_5C_{11}+C_{16})\delta+C_4C_5(C_{15}+C_{18})\delta^{1.5}+C_4C_5(C_{14}+C_{17})\delta^2+C_4C_5C_{13}\delta^{2.5}\Big).$$

Let

$$K_1\triangleq C_1,\quad K_2\triangleq C_4C_8+C_4C_5C_{10}+C_6+C_{12}.$$

Then finally we obtain

$$\|\hat{x}_\delta(t) - x^*(t)\| \le K_2 e^{K_1 T}\sqrt{\delta} + o(\sqrt{\delta}), \quad t \in [0,T]. \qquad \square$$

Let us now describe a method for constructing piecewise constant weak L^2-approximations of $u_\delta(\cdot)$ that satisfy the nonconvex constraints (3.2). We will use the auxiliary weak L^2-approximations $\hat{u}_\delta(\cdot)$ (4.1), which satisfy the convex constraints (2.2).

The construction is based on Carathéodory's theorem on the structure of a convex set (see Theorem 17.1 in [18], Ch. IV).

Let us fix δ and consider the auxiliary approximation $\hat{u}_\delta(\cdot)$. By Carathéodory's theorem, for each ith interval $[t_i, t_{i+1}]$ and the corresponding value of the auxiliary approximation $\hat{u}_{\delta,i}$, there exists a convex combination of elements $\{\bar{u}_{i,k},\ k = 1,\ldots,m+1\}$ of the set $\mathbf{U}$ such that

$$\begin{gathered}\hat{u}_{\delta,i} = \sum_{k=1}^{m+1} \lambda_{i,k}\,\bar{u}_{i,k},\\ \lambda_{i,1} + \lambda_{i,2} + \ldots + \lambda_{i,m+1} = 1, \quad 0 \le \lambda_{i,k} \le 1, \quad k = 1,\ldots,m+1.\end{gathered} \tag{4.13}$$

Based on the coefficients of combination (4.13), we can construct an additional nonuniform partition of each interval $[t_i, t_{i+1}]$ into subintervals of lengths $h\lambda_{i,1}$, $h\lambda_{i,2}$, …, $h\lambda_{i,m+1}$. For brevity, we write these subintervals as $\Lambda_{i,k}$.

Assume that, on each (i,k)th approximation interval,

$$u_\delta(t) = \bar{u}_{i,k}, \quad t \in \Lambda_{i,k}, \quad i = 0,\ldots,N-1, \quad k = 1,\ldots,m+1. \tag{4.14}$$

Thus, each auxiliary control $\hat{u}_\delta(\cdot)$ is associated with the approximation $u_\delta(\cdot)$ constructed by the described method (4.13), (4.14). Let us show that the approximations $u_\delta(\cdot)$ satisfy the requirements of the CRP (3.7)–(3.9).

Observe the following property of the approximations:

$$\int_{t_i}^{t_{i+1}} (u_\delta(t) - \hat{u}_\delta(t))dt = 0, \quad i = 0,\ldots,N-1. \tag{4.15}$$

Indeed, by construction (4.13),

$$\int_{t_i}^{t_{i+1}} (u_\delta(t) - \hat{u}_\delta(t))dt = \sum_{k=1}^{m+1} \int_{\Lambda_{i,k}} (\bar{u}_{i,k} - \hat{u}_{\delta,i})dt = \sum_{k=1}^{m+1} [h\lambda_{i,j}\bar{u}_{i,k}] - h\hat{u}_{\delta,i} = 0.$$

The following theorem establishes the validity of condition (3.9) of the CRP; i.e., it states that the constructed approximations $u_\delta(\cdot)$ converge weakly in L^2 to the normal control.

Theorem 2. *Suppose that approximations $u_\delta(\cdot)$ are constructed from the auxiliary approximations $\hat{u}_\delta(\cdot)$ (4.1) by the described method (4.13), (4.14).*

Then condition (3.9) of the weak L^2-convergence of the controls $u_\delta(\cdot) \to u^*(\cdot)$ is satisfied.

Proof. First, we establish the convergence (3.6):

$$\int_0^T \langle \xi(t), u_\delta(t) - \hat{u}_\delta(t)\rangle dt \overset{\delta\to 0}{\longrightarrow} 0 \ \ \forall \xi(\cdot) \in C([0,T], \mathbb{R}^m).$$

Expand the integral:

$$\Big|\int_0^T \langle \xi(t), u_\delta(t)-\hat u_\delta(t)\rangle dt\Big| = \Big|\sum_{i=0}^{N-1}\Big[\int_{t_i}^{t_{i+1}}\langle \xi(t)-\xi(t_i)+\xi(t_i), u_\delta(t)-\hat u_\delta(t)\rangle dt\Big]\Big|$$
$$= \Big|\sum_{i=0}^{N-1}\Big\{\int_{t_i}^{t_{i+1}}\langle \xi(t)-\xi(t_i), u_\delta(t)-\hat u_\delta(t)\rangle dt + \Big\langle \xi(t_i), \Big[\int_{t_i}^{t_{i+1}}(u_\delta(t)-\hat u_\delta(t))dt\Big]\Big\rangle\Big\}\Big|. \tag{4.16}$$

By property (4.15) of the approximations, the integrals in square brackets in the last line of (4.16) are zero.

Let us estimate the increment of the continuous function $\xi(\cdot)$ in terms of its modulus of continuity $\omega_\xi(\cdot)$:

$$\|\xi(t)-\xi(t_i)\| \le \omega_\xi(h). \tag{4.17}$$

Substitute estimate (4.17) into (4.16):

$$\Big|\int_0^T \langle \xi(t), u_\delta(t)-\hat u_\delta(t)\rangle dt\Big| \le \sum_{i=0}^{N-1} h\omega_\xi(h)2R_U \le 2T\omega_\xi(h)R_U \xrightarrow{h\to 0} 0,$$

since $N=\lceil T/h\rceil$.

Convergence (3.6) is proved.

As follows from Assertion 1, in this case there is also the weak convergence in the space L^2:

$$\int_0^T \langle \varphi(t), u_\delta(t)-\hat u_\delta(t)\rangle dt \xrightarrow{\delta\to 0} 0 \quad \forall\varphi(\cdot)\in L^2([0,T],\mathbb{R}^m). \tag{4.18}$$

However, it is assumed that, by condition (3.9) of the CRP, the auxiliary approximations $\hat u_\delta(\cdot)$ (4.1) themselves converge weakly in L^2 to $u^*(\cdot)$. Hence, it follows from (4.18) that the approximations $u_\delta(\cdot)$ weakly converge to $u^*(\cdot)$. □

Let us now show that condition (3.8) of convergence of the trajectories is satisfied and obtain an estimate for the discrepancy of trajectories.

Lemma 2. *Let approximations $u_\delta(\cdot)$ be constructed based on the auxiliary approximations $\hat u_\delta(\cdot)$ (4.1) by the described method (4.13), (4.14). Further, let $x_\delta(\cdot)$ and $\hat x_\delta(\cdot)$ be the trajectories of system (2.1) generated by the approximations $u_\delta(\cdot)$ and $\hat u_\delta(\cdot)$, respectively, under identical boundary conditions.*

Then there exist constants K_3 and K_4, depending on the properties of the functions $G(t,x)$ and $f(t,x)$ from the dynamics (2.1), such that

$$\|x_\delta(t)-\hat x_\delta(t)\| \le \big(K_3\omega_G(h)+K_4h\big)e^{K_1T}, \quad t\in[0,T],$$

where $\omega_G(\cdot)$ is the modulus of continuity of the function $G(\cdot)$.

Proof. Consider the discrepancy

$$
\begin{aligned}
\|x_\delta(t)-\hat{x}_\delta(t)\| &= \Bigg\| \int\limits_0^t \Big[G(\tau,x_\delta(\tau))u_\delta(\tau) - G(\tau,\hat{x}_\delta(\tau))\hat{u}_\delta(\tau)\big[\pm G(\tau,\hat{x}_\delta(\tau))u_\delta(\tau)\big] \\
&\quad + f(\tau,x_\delta(\tau)) - f(\tau,\hat{x}_\delta(\tau))\Big]d\tau\Bigg\| \le \Bigg\|\int\limits_0^t \big[G(\tau,\hat{x}_\delta(\tau))(u_\delta(\tau)-\hat{u}_\delta(\tau))\big]d\tau\Bigg\| \\
&\quad + \Bigg\|\int\limits_0^t \big[(G(\tau,x_\delta(\tau)) - G(\tau,\hat{x}_\delta(\tau)))u_\delta(\tau)\big]d\tau\Bigg\| \\
&\quad + \Bigg\|\int\limits_0^t \big[f(\tau,x_\delta(\tau)) - f(\tau,\hat{x}_\delta(\tau))\big]d\tau\Bigg\| \triangleq \mathbf{A_t}+\mathbf{B_t}+\mathbf{C_t}.
\end{aligned}
\tag{4.19}
$$

I. Let us estimate the first term $\mathbf{A_t}$ in expression (4.19).

We will assume that the auxiliary approximations $\hat{u}_\delta(\cdot)$ satisfy condition (3.2), as well as conditions (3.8) and (3.9) of the CRP. Then, first, $\|\hat{u}_\delta(t)\| \le R_U$, and, second, by the uniform convergence of the trajectories (3.8) (condition (3.8) of the CRP), there exists a value $\hat{\delta}_0 \in (0,\delta_0]$ such that, for $\delta \in (0,\hat{\delta}_0]$,

$$
\hat{x}_\delta(t) \in \Psi \implies \|G(t,\hat{x}_\delta(t))\|_2 \le R_G, \quad \|f(t,\hat{x}_\delta(t))\| \le R_f, \quad t\in[0,T],
$$

according to the definitions (3.5). The compact set Ψ is defined in Assumption 1 (Subsect. 2.3, formula (2.4)).

For any time $t\in[0,T]$, we can choose an index $j=j(t)$ such that $t\in[t_j,t_{j+1}]$ and

$$
\begin{aligned}
\mathbf{A_t} &= \Bigg\|\int\limits_0^t \big[G(\tau,\hat{x}_\delta(\tau))(u_\delta(\tau)-\hat{u}_\delta(\tau))\big]d\tau\Bigg\| \\
&\le \Bigg\|\sum_{i=0}^{j-1}\Bigg\{\int\limits_{t_i}^{t_{i+1}} \Big[\big(G(\tau,\hat{x}_\delta(\tau))\big[\pm G(t_i,\hat{x}_\delta(t_i))\big]\big)(u_\delta(\tau)-\hat{u}_\delta(\tau))\Big]d\tau\Bigg\}\Bigg\| \\
&\quad + \Bigg\|\int\limits_{t_j}^{t}\big[G(\tau,\hat{x}_\delta(\tau))(u_\delta(\tau)-\hat{u}_\delta(\tau))\big]d\tau\Bigg\| \\
&\le \sum_{i=0}^{j-1}\Bigg\{\int\limits_{t_i}^{t_{i+1}}\big[\,\|G(\tau,\hat{x}_\delta(\tau)) - G(t_i,\hat{x}_\delta(t_i))\|\,\|u_\delta(\tau)-\hat{u}_\delta(\tau)\|\,\big]d\tau \\
&\quad + \Bigg\|G(t_i,\hat{x}_\delta(t_i))\Bigg[\int\limits_{t_i}^{t_{i+1}}(u_\delta(\tau)-\hat{u}_\delta(\tau))d\tau\Bigg]\Bigg\|\Bigg\} + hR_G 2R_U.
\end{aligned}
\tag{4.20}
$$

By property (4.15) of the approximations, the integrals in square brackets in the last line of (4.20) are zero.

By properties (2.5) of the matrix $G(\cdot)$,

$$\|G(\tau,\hat{x}_\delta(\tau)) - G(t_i,\hat{x}_\delta(t_i))\| \le \omega_G(h) + L\|\hat{x}_\delta(\tau) - \hat{x}_\delta(t_i)\|, \tag{4.21}$$

where $\omega_G(\cdot)$ is the modulus of continuity of the function $\|G(\cdot)\|$.

For $\tau \in [t_i, t_{i+1}]$, we have the inequality

$$\|\hat{x}_\delta(\tau) - \hat{x}_\delta(t_i)\| \le \int\limits_{t_i}^{\tau} \|G(\theta,\hat{x}_\delta(\theta))\hat{u}_\delta(\theta) + f(\theta,\hat{x}_\delta(\theta))\|\,d\theta \le h(R_G R_U + R_f).$$

Hence, returning to inequality (4.21), we get

$$\|G(\tau,\hat{x}_\delta(\tau)) - G(t_i,\hat{x}_\delta(t_i))\| \le \omega_G(h) + h\,L(R_G R_U + R_f), \quad \tau \in [t_i, t_{i+1}]. \tag{4.22}$$

Substitute estimate (4.22) into (4.20):

$$\begin{aligned}\mathbf{A_t} &\le \sum_{i=1}^{N-1} \big[h\big(\omega_G(h) + h\,L(R_G R_U + R_f)\big)2R_U\big] + h\,2R_G R_U \\ &\le 2R_U T\big(\omega_G(h) + h\,L(R_G R_U + R_f)\big) + h\,2R_G R_U,\end{aligned} \tag{4.23}$$

since $N = \lceil T/h \rceil$.

II. Estimate the term $\mathbf{B_t}$ from expression (4.19). Since the matrix function $G(\cdot)$ is Lipschitz (2.5), we obtain

$$\mathbf{B_t} = \left\| \int\limits_0^t \big[\big(G(\tau,x_\delta(\tau)) - G(\tau,\hat{x}_\delta(\tau))\big)u_\delta(\tau)\big]\,d\tau \right\| \le \int\limits_0^t \big[L\|x_\delta(\tau) - \hat{x}_\delta(\tau)\|\,\|u_\delta(\tau)\|\big]\,d\tau. \tag{4.24}$$

III. Estimate the term $\mathbf{C_t}$ from expression (4.19). Since the vector function $f(\cdot)$ is Lipschitz (2.5), we obtain

$$\mathbf{C_t} = \left\| \int\limits_0^t \big[f(\tau,x_\delta(\tau)) - f(\tau,\hat{x}_\delta(\tau))\big]\,d\tau \right\| \le \int\limits_0^t L\|x_\delta(\tau) - \hat{x}_\delta(\tau)\|d\tau. \tag{4.25}$$

Finally, substituting the estimates for $\mathbf{A_t}$ (4.23), $\mathbf{B_t}$ (4.24), and $\mathbf{C_t}$ (4.25) into the discrepancy estimate (4.19), we obtain

$$\begin{aligned}\|x_\delta(t) - \hat{x}_\delta(t)\| &\le 2R_U T\big(\omega_G(h) + h\,L(R_G R_U + R_f)\big) \\ &+ h\,2R_G R_U + \int\limits_0^t [L\|x_\delta(\tau) - \hat{x}_\delta(\tau)\|(\|u_\delta(\tau)\| + 1)]\,d\tau.\end{aligned}$$

Then, by the Gronwall–Bellman lemma,

$$\begin{aligned}\|x_\delta(t) - \hat{x}_\delta(t)\| &\le \Big(2R_U T\omega_G(h) + \big(2R_U T L(R_G R_U + R_f) + 2R_G R_U\big)h\Big)e^{LT(R_U+1)} \\ &= \big(K_3\omega_G(h) + K_4 h\big)e^{K_1 T} \xrightarrow{h\to 0} 0,\end{aligned}$$

$$K_3 \triangleq 2R_U T, \quad K_4 \triangleq 2R_U T L(R_G R_U + R_f) + 2R_G R_U. \qquad \square$$

The validity of Theorem 3 follows from Lemmas 1 and 2.

Theorem 3. *Suppose that Assumptions 1–5 are satisfied. Let $x_\delta(\cdot)$ be the trajectories of system (2.1) generated by the approximations $u_\delta(\cdot)$ (4.13), (4.14). Assume that the parameters $\delta \le \delta_0$, $h = h(\delta) \le h_0$, and $\alpha = \alpha(\delta) > 0$ tend to zero and are matched as in (4.3). Then*

$$\|x_\delta(\cdot) - x^*(\cdot)\|_C \xrightarrow{\delta \to 0} 0.$$

If the parameters h and α are chosen according to formulas (4.5), then there exist constants K_1, K_3, and K_5, depending on the properties of the functions $G(t,x)$ and $f(t,x)$ from the dynamics (2.1), such that

$$\|x_\delta(t) - x^*(t)\| \le e^{K_1 T}(K_3 \omega_G(\sqrt{\delta}) + K_5\sqrt{\delta}) + o(\delta).$$

Theorem 3 says that the CRP condition (3.8) is satisfied; i.e. the trajectories generated by the approximating controls uniformly converge to the observed trajectory. Thus, it is shown that the constructed approximations $u_\delta(\cdot)$ satisfy all conditions of the CRP.

CONCLUSIONS

The control reconstruction problem has been considered for dynamic deterministic affine-control systems in the case of nonconvex geometric constraints on the controls. Sliding controls are allowed. In the problem, it is required to reconstruct an unknown control that generates the observed trajectory based on inaccurate discrete measurements of this trajectory.

The notion of normal control is introduced. This is a measurable control that generates the observed trajectory and is defined uniquely.

It is shown that the convergence of approximations of the normal control in the weak topology of the space L^2 should be used in the case under consideration.

A well-posed problem of reconstruction of the normal control is posed, and its solution is proposed and justified.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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