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FORMATIONS OF FINITE GROUPS IN POLYNOMIAL TIME II: THE $\mathfrak{F}$-HYPERCENTER AND ITS GENERALIZATIONS[1]

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For a wide family of formations $\mathfrak{F}$ (which includes Baer-local formations) of finite groups it is proved that the $\mathfrak{F}$-hypercenter of a permutation finite group of degree n can be computed in polynomial time in n. In particular, the algorithms for computing the $\mathfrak{F}$-hypercenter for the following classes of groups are suggested: hereditary local formations with the Shemetkov property, rank formations, formations of all quasinilpotent, Sylow tower of type φ, p-nilpotent, supersoluble, w-supersoluble and SC-groups. For some of these formations $\mathfrak{F}$ algorithms for the computation of the intersection of all maximal $\mathfrak{F}$-subgroups of a finite group are suggested.

Keywords: Finite group; $\mathfrak{F}$-hypercenter; Baer-local formation; permutation group computation; polynomial time algorithm.

В. И. Мурашко. Распознавание формаций конечных групп за полиномиальное время II: $\mathfrak{F}$-гиперцентр и его обобщения

Для широкого семейства формаций $\mathfrak{F}$ (включающего в себя композиционные формации) конечных групп доказано, что $\mathfrak{F}$-гиперцентр конечной группы перестановок степени n может быть вычислен за полиномиальное время от n. В частности, предложены алгоритмы вычисления $\mathfrak{F}$-гиперцентра для следующих классов групп: наследственные локальные формации с условием Шеметкова, ранговые формации, формации всех квазинильпотентных, φ-дисперсивных, p-нильпотентных, сверхразрешимых, w-сверхразрешимых и SC-групп. Для некоторых из этих формаций $\mathfrak{F}$ предложены алгоритмы вычисления пересечения всех максимальных $\mathfrak{F}$-подгрупп конечной группы.

Ключевые слова: Конечная группа; $\mathfrak{F}$-гиперцентр; композиционная формация; вычисления в группах перестановок; полиномиальный алгоритм.

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1. Introduction

All groups considered here are finite. Recall that a class of groups is a collection $\mathfrak{X}$ of groups with the property that if $G \in \mathfrak{X}$ and if $H \simeq G$, then $H \in \mathfrak{X}$. The theory of classes of groups is well developed nowadays (for example, see [1–4]) and has various applications (for example, in the theory of formal languages [5], in the solution of Yang-Baxter equation [6] and etc.) The main its tasks are to construct classes of groups and to recognize is a given group belongs to a given class or not. With a class of groups $\mathfrak{X}$ one can associate the canonical subgroups such as

— the $\mathfrak{X}$-residual, i.e. the smallest normal subgroup of a group such that the quotient group over it belongs to $\mathfrak{X}$;

— the $\mathfrak{X}$-radical, i.e. the greatest normal $\mathfrak{X}$-subgroup;

— an $\mathfrak{X}$-projector, i.e. an $\mathfrak{X}$-maximal subgroup that keeps this property in every quotient group;

— an $\mathfrak{X}$-injector, i.e. an $\mathfrak{X}$-maximal subgroup which intersects with every normal subgroup by its $\mathfrak{X}$-maximal subgroup;

— the $\mathfrak{X}$-hypercenter, i.e. the greatest normal subgroup of G such that all chief factors of G below it are $\mathfrak{X}$-central.

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Even if a group does not belong to $\mathfrak{X}$, then these subgroups encode some of its properties associated with $\mathfrak{X}$.

The algorithms for computing the $\mathfrak{X}$-residual and $\mathfrak{X}$-projectors (of a soluble group) were presented in [7;8]. The computation of the $\mathfrak{X}$-residual of not necessary soluble groups was discussed in [9]. The algorithms for computing the $\mathfrak{X}$-radical and $\mathfrak{X}$-injectors (of a soluble group) were presented in [8].

Nevertheless the algorithms for the computation of the $\mathfrak{F}$-hypercenter were not suggested before for a formation $\mathfrak{F}$. Note that the $\mathfrak{F}$-hypercenter and its generalizations play an important role in the theory of formations: they encode a significant information about the structure of a group associated with a formation $\mathfrak{F}$ (for example, see chapters of monographs [2, Chapter IV(6)], [3, Chapter 1] and [4, §14], and papers [10–14]). That is why it is important to have the effective algorithms for the computation of the $\mathfrak{F}$-hypercenter of a given group.

A group can be represented in different ways. In this paper we will consider only permutation groups because the computational theory of permutation groups is well developed (see [15]). *The aim of this paper is to find effective algorithms* (which runs in polynomial time for permutation groups) *for the computation of the $\mathfrak{F}$-hypercenter for a wide family of formations $\mathfrak{F}$ of not necessary soluble groups*. We leave the reader to decide how our algorithms can be applied in the case of non-permutation groups.

2. The Main Result

Recall that the hypercenter of a group is just the final member of its upper central series. The formational generalizations of the hypercenter were developed in the papers of Baer [16], Huppert [17], Shemetkov [18] and appeared in the final form in [4] (see also [3, Chapter 1]). Let $\mathfrak{X}$ be a class of groups. A chief factor H/K of G is called $\mathfrak{X}$-*central* in G provided that the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ of H/K with $G/C_G(H/K)$ corresponding to the action by conjugation of G on H/K belongs to $\mathfrak{X}$ (see [4, p. 127–128]). The $\mathfrak{X}$-hypercenter $\mathrm{Z}_{\mathfrak{X}}(G)$ is the greatest normal subgroup of G such that all chief factors of G below it are $\mathfrak{X}$-central.

From Barnes–Kegel Theorem [2, IV, Proposition 1.5] it follows that if $\mathfrak{F}$ is a formation and $G \in \mathfrak{F}$, then $G = \mathrm{Z}_{\mathfrak{F}}(G)$. The converse of this statement is false as we can see on the example of the class of all abelian groups. Shemetkov [19] asked to described all formations $\mathfrak{F} = (G \mid G = \mathrm{Z}_{\mathfrak{F}}(G))$ (such formations are called Z-saturated [20]). The solution of this problem was started in [20; 21]. The method for calculating the $\mathfrak{F}$-hypercenter of a given group may be a useful tool in the solution of this problem.

In [22] the example of a permutation group of degree n was given such that it has a quotient with no faithful representations of degree less than $2^{n/4}$. That is why the given definition of the $\mathfrak{X}$-central chief factor is not good from the computational point of view. Therefore we will consider a more general definition:

Definition 1 [9, Definition 2]. Let f be a function which assigns 0 or 1 to every group G and its chief factor H/K such that

(1) $\mathrm{f}(H/K, G) = \mathrm{f}(M/N, G)$ whenever H/K and M/N are G-isomorphic chief factors of G;

(2) $\mathrm{f}(H/K, G) = \mathrm{f}((H/N)/(K/N), G/N)$ for every $N \trianglelefteq G$ with $N \leq K$.

Such functions f will be called chief factor functions. Let

$$\mathcal{C}(\mathrm{f}) = (G \mid G \simeq 1 \text{ or } \mathrm{f}(H/K, G) = 1 \text{ for every chief factor } H/K \text{ of the group } G).$$

The class $\mathcal{C}(\mathrm{f})$ is a formation [9, Lemma 3].

In particular if $\mathrm{f}(H/K, G) = 1$ iff H/K is $\mathfrak{X}$-central, then f is a chief factor function. So Z-saturated formations (and hence local and Baer-local formations) are the particular cases of this construction. With each chief factor function we can associate the following subgroup:

Definition 2 [9, Definition 2]. Denote by $\mathrm{Z}(G, \mathrm{f})$ the greatest normal subgroup of G such that $\mathrm{f}(H/K, G) = 1$ for every chief factor H/K of G below it.

The subgroup $\mathrm{Z}(G,\mathrm{f})$ becomes the $\mathfrak{X}$-hypercenter of G with the right choice of f. In particular, if f checks if H/K is central (resp. cyclic) in G, then $\mathrm{Z}(G,\mathrm{f})$ is the (resp. supersoluble) hypercenter.

Example 1. Let f_1 and f_2 check if H/K is abelian and if $G/C_G(H/K)$ is soluble respectively. Then $\mathcal{C}(\mathrm{f}_1)=\mathcal{C}(\mathrm{f}_2)=\mathfrak{S}$ is the class of all soluble groups. Note that $\mathrm{Z}(G,\mathrm{f}_1)$ is the soluble radical of G and $\mathrm{Z}(G,\mathrm{f}_2)$ is the soluble hypercenter of G. If we consider the semidirect product of the alternating group A_5 of degree 5 and its faithful simple module over $\mathbb{F}_5$ (it exists by [2, B, Theorem 10.3]), then we will see that these subgroups are different.

The main result of the paper is

Theorem 1. *Assume that* $\mathrm{f}(H/T,G)$ *can be computed in polynomial time* (*in* n) *for every permutation group* G *of degree* n *and its chief factor* H/T. *Then the subgroup* $\mathrm{Z}(G/K,\mathrm{f})$ *is well defined and can be computed in polynomial time* (*in* n) *for every permutation group* G *of degree* n *and its normal subgroup* K *by Algorithm* 1.

In Section 5 we will discuss how this theorem can be applied to various formations.

3. Preliminaries

All unexplained notations and terminologies are standard. The reader is referred to [1–3] if necessary. Recall that $\langle X\rangle$ denotes the subgroup generated by X; $\mathrm{Z}(G)$ denotes the center of G; $\mathrm{O}_p(G)$ is the greatest normal p-subgroup of G; $\mathrm{O}^\pi(G)$ is the smallest normal subgroup of G of π-index; $\Omega_1(G)$ denotes the subgroup that is generated by elements of order p for a p-group G; S_n denotes the symmetric group of degree n; $\mathfrak{G}_\pi$ denotes the class of all π-groups.

Recall that a *formation* is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images (i.e. from $G\in\mathfrak{F}$ and $N\trianglelefteq G$ it follows that $G/N\in\mathfrak{F}$) and subdirect products (i.e. from $G/N_1\in\mathfrak{F}$ and $G/N_2\in\mathfrak{F}$ it follows that $G/(N_1\cap N_2)\in\mathfrak{F}$). If $\mathfrak{F}$ is a non-empty formation, then in every group G there exists the $\mathfrak{F}$-residual, i.e. the smallest normal subgroup of G with $G/G^{\mathfrak{F}}\in\mathfrak{F}$.

We use standard computational conventions of abstract finite groups equipped with polynomial-time procedures to compute products and inverses of elements (see [15, Chapter 2]). For both input and output, groups are specified by generators. We will consider only $G=\langle S\rangle\leq S_n$ with $|S|\leq n^2$. If necessary, Sims' algorithm [15, Parts 4.1 and 4.2] can be used to arrange that $|S|\leq n^2$. Quotient groups are specified by generators of a group and its normal subgroup. *For the rest of the paper n is used to denote the degree of the input permutations. A polynomial-time algorithm is an algorithm whose running time is upper-bounded by some polynomial function of n.* We need the following well known basic tools in our proofs (see, for example [23] or [15, pp. 49–50]).

List of Some Known Algorithms. *Given normal subgroups A and B of a permutation group G of degree n with $A\leq B$, in polynomial time in n one can solve the following problems*:

(*A*1) *Find the centralizer $C_{G/A}(B/A)$ of B/A in G/A* [23, P6(i)].

(*A*2) *Find the soluble residual $(G/A)^{\mathfrak{S}}$ of G/A* [23, P14(i)].

(*A*3) *Find the order $|G/A|$ of G/A* [23, P1].

(*A*4) *Find* $\mathrm{Z}(G/A)$ [23, P14(iii)], $\mathrm{O}_p(G/A)$ *and* $\mathrm{O}^\pi(G/A)$ [23, P16].

(*A*5) *Find a chief series for G containing A and B* [23, P11].

(*A*6) *Given $H=\langle S_1\rangle, K=\langle S_2\rangle\leq G$ find $\langle H,K\rangle=\langle S_1,S_2\rangle$ and $[H,K]=\langle[s_1,s_2]\mid s_1\in S_1, s_2\in S_2\rangle^{\langle H,K\rangle}$* [23, P3(i)].

(*A*7) *Check if G/A is simple* [23, P10(i)].

Note that (*A*1), (*A*4), (*A*5) and (*A*7) are obtained with the help of Classification of Finite Simple Groups. The following lemma plays an important role in our proves.

Lemma 1 [24]. *If $G\leq S_n$ for $n\geq 2$, then the length of every subgroup chain in G is at most* $2n-3$.

Algorithm 1: GFHYPERCENTER(G, K, f)

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Result: Z(G/K, f).
Data: A normal subgroup K of a group G and a chief factor function f.
Z ← K;
Compute a chief series K = G_0 ⊴ G_1 ⊴ ··· ⊴ G_m = G of G;
for i ∈ {1, ..., m} do
  if f(G_i/G_{i-1}, G) = 1 then
    if |π(G_i/G_{i-1})| > 1 then
      C/Z ← C_{G_i/Z}(G_{i-1}/Z)^𝔖;
      if |C/Z| = |G_i/G_{i-1}| then
        Z ← C;
      end
    else
      p ← π(G_i/G_{i-1})[1];
      P/Z ← Ω_1(Z(O_p(G_i/Z)));
      For each generator g of G find the linear transformation which this element
       induces on P/Z;
      Let R be the algebra generated by above mentioned transformations.
       Decompose the R-module P/Z into the sum P_1 ⊕ ··· ⊕ P_k of indecomposable
       submodules;
      for j ∈ {1, ..., k} do
        if P_j is a simple module then
          Find the corresponding to P_j subgroup P_j/Z;
          if ⟨P_j/Z, G_{i-1}/Z⟩ = G_i/Z then
            Z ← P_j;
            Leave the current for cycle;
          end
        end
      end
    end
  end
end
return Z/K
```

4. Proof of Theorem 1

The proof of Theorem 1 is based on the following 5 lemmas.

Lemma 2. *Let* f *be a chief factor function. The following statements hold:*

(1) $\mathrm{Z}(G, \mathrm{f})$ *is well defined for any group* G.

(2) *Let* $Z/K = \mathrm{Z}(G/K, \mathrm{f})$. *Then* Z *is the greatest normal subgroup of* G *such that it contains* K *and* $\mathrm{f}(H/T, G) = 1$ *for every chief factor* H/T *of* G *with* $K \leq T \leq H \leq Z$.

Proof. (1) Let M and N be normal subgroups of a group G. Then form [2, A, The Isomorphism Theorems(b)] it follows that every chief factor of G below MN is G-isomorphic to a chief factor of G below either M or N. So if $\mathrm{f}(H/K, G) = 1$ for every chief factor H/K of G below M and N, then $\mathrm{f}(H/K, G) = 1$ for every chief factor H/K of G below MN by (1) of Definition 1. It means that the subgroup $\mathrm{Z}(G, \mathrm{f})$ is well defined.

(2) Let Z be the greatest normal subgroup of G such that it contains K and $\mathrm{f}(H/T, G) = 1$ for every chief factor H/T of G with $K \leq T \leq H \leq Z$. Now $\mathrm{f}((H/K)/(T/K), G/K) = \mathrm{f}(H/T, G) = 1$

for every chief factor H/T of G such that $(H/K)/(T/K)$ is a chief factor of G/K below Z/K by (2) of Definition 1. Hence $Z/K \leq \mathrm{Z}(G/K, \mathrm{f})$. Assume that $\mathrm{f}((M/K)/(Z/K), G/K) = 1$ for some chief factor $(M/K)/(Z/K)$ of G/K. Then $\mathrm{f}(M/Z, G) = 1$ for some chief factor M/Z of G by (2) of Definition 1, a contradiction with the definition of Z. Thus $Z/K = \mathrm{Z}(G/K, \mathrm{f})$. □

Lemma 3. *Let H/T be a chief factor of G, $K \trianglelefteq G$, $K \leq T$, $Z_1/K = \mathrm{Z}(G/K, \mathrm{f}) \cap T/K$ and $Z_2/K = \mathrm{Z}(G/K, \mathrm{f}) \cap H/K$. Then $Z_2/Z_1 \not\simeq 1$ if and only if $\mathrm{f}(H/T, G) = 1$ and $H/Z_1 = T/Z_1 \times Z/Z_1$ for some $Z \trianglelefteq G$. In this case $Z_2 = Z$.*

Proof. Note that $Z_1/K = Z_2/K \cap T/K = (Z_2 \cap T)/K$. Hence $Z_1 = Z_2 \cap T$. Now

$$Z_2/Z_1 = Z_2/(Z_2 \cap T) \simeq Z_2T/T \leq H/T.$$

Hence Z_2/Z_1 is G-isomorphic to either 1 or H/T.

Assume that $Z_2/Z_1 \not\simeq 1$. It means that $1 = \mathrm{f}(Z_2/Z_1, G) = \mathrm{f}(H/T, G)$ by (2) of Lemma 2 and Z_2/Z_1 is a minimal normal subgroup of G/Z_1 below H/Z_1 and not contained in T/Z_1. Hence $H/Z_1 = T/Z_1 \times Z_2/Z_1$.

Assume now that $\mathrm{f}(H/T, G) = 1$ and $H/Z_1 = T/Z_1 \times Z/Z_1$ for some $Z \trianglelefteq G$. Now $\mathrm{f}(A/B, G) = 1$ for every chief factor A/B of G between Z_1 and K by (2) of Lemma 2. From $H/T = ZT/T \simeq Z/(Z \cap T) = Z/Z_1$ it follows that $\mathrm{f}(Z/Z_1, G) = 1$. It means that

$$Z_1/K < Z/K \leq \mathrm{Z}(G/K, \mathrm{f}) \cap H/K = Z_2/K.$$

Therefore $Z_2/Z_1 \not\simeq 1$ and $Z = Z_2$. □

Lemma 4. *In the notations of Theorem 1 and Lemma 3 if H/T is non-abelian and Z_1 is given, then we can compute Z_2 in polynomial time in n.*

Proof. Assume that H/T is non-abelian. If $\mathrm{f}(H/T, G) = 0$, then $Z_1 = Z_2$ by Lemma 3. So we can assume that $\mathrm{f}(H/T, G) = 1$. Let $C/Z_1 = C_{H/Z_1}(T/Z_1)^{\mathfrak{S}}$. We claim that $Z_2/Z_1 \not\simeq 1$ iff $|C/Z_1| = |H/T|$. In this case if $Z_2/Z_1 \not\simeq 1$, then $Z_2/Z_1 = C/Z_1$.

Suppose that $Z_2/Z_1 \not\simeq 1$. From Lemma 3 it follows that $H/Z_1 = T/Z_1 \times Z/Z_1$ for some $Z \trianglelefteq G$. Then $Z/Z_1 \leq C_{H/Z_1}(T/Z_1)$. Hence

$$C_{H/Z_1}(T/Z_1) = TZ/Z_1 \cap C_{H/Z_1}(T/Z_1) = (C_{H/Z_1}(T/Z_1) \cap T/Z_1)(Z/Z_1) = \mathrm{Z}(T/Z_1) \times (Z/Z_1).$$

Recall that $H/T \simeq Z/Z_1$ is a direct product of simple non-abelian groups. It means that

$$C_{H/Z_1}(T/Z_1)^{\mathfrak{S}} = Z/Z_1 = Z_2/Z_1$$

by Lemma 3. Thus $|C/Z_1| = |H/T|$.

Suppose that $|C/Z_1| = |H/T|$. Note that C/Z_1 is non-abelian (because it is the soluble residual) and $C/Z_1 \cap T/Z_1 \leq \mathrm{Z}(T/Z_1)$ is abelian. Therefore $C/Z_1 \not\leq T/Z_1$. From

$$C/Z_1 \text{ char } C_{H/Z_1}(T/Z_1) = C_{G/Z_1}(T/Z_1) \cap H/Z_1 \trianglelefteq G/Z_1$$

it follows that $C/Z_1 \trianglelefteq G$. It means that $H = TC$ and $H/T = CT/T \simeq C/(C \cap T)$. From $|C/Z_1| = |H/T|$ it follows that $|C \cap T| = |Z_1|$. Since $Z_1 \leq C \cap T$, we see that $C \cap T = Z_1$. It means that $H/Z_1 = T/Z_1 \times C/Z_1$. Thus $C/Z_1 = Z_2/Z_1$ by Lemma 3.

By the statement of Theorem 1 we can compute $\mathrm{f}(H/T, G)$ in polynomial time. Now $C/Z_1 = C_{H/Z_1}(T/Z_1)$ can be computed in polynomial time by $(A1)$. We can compute $(C/Z_1)^{\mathfrak{S}}$ by $(A2)$. With the help of $(A3)$ we can check if $|H/T| = |C/Z_1|$ in polynomial time. □

Lemma 5. *If K is a normal subgroup of a permutation group G of degree n such that G/K is an abelian p-group for some prime p, then $\Omega_1(G/K)$ can be computed in polynomial time in n.*

Proof. Recall that every abelian group can be presented as a direct product of its cyclic subgroups (for example, see [2, A, Theorem 4.18]). Assume that $G/K = \langle c_1 K\rangle \times \cdots \times \langle c_t K\rangle$ is such presentation. By Lemma 1 we have that $t \leq 2n$. Note that if the order of a permutation of degree n is a power of a prime, then it is not greater than n. It means that the order of every $c_i K$ is not greater than n. If cK is an element of order p in G/K, then it can be expressed uniquely in the form $\prod_{i=1}^{t} c_i^{a_i} K$ where a_i is less than the order of $c_i K$. From $c^p \in K$ it follows that $c_i^{a_i p} \in K$ for every i. Hence $c_i^{a_i} K$ belongs to $P_i/K = \langle (c_i K)^{\frac{|\langle c_i, K\rangle|}{p\cdot|K|}}\rangle$ (here and later in the generation of subgroups, i.e. $\langle \dots \rangle$, we use only the generating set of K). From $\frac{|\langle c_i, K\rangle|}{p\cdot|K|} \leq n$ it follows that $(c_i K)^{\frac{|\langle c_i, K\rangle|}{p\cdot|K|}}$ can be computed in polynomial time by $(A3)$. From the other hand every element from $P/K = \prod_{i=1}^{t} P_i/K$ has order p. Thus $P/K = \Omega_1(G/K)$. So if the presentation $G/K = \langle c_1 K\rangle \times \cdots \times \langle c_t K\rangle$ is given, then $\Omega_1(G/K)$ can be computed in polynomial time by $(A6)$.

In the subsequent proof we need to use some algorithms with complexity in NC ("Nick's Class"). For our proof it is enough to know that problems solved in NC can be solved in polynomial time (for example, see [25, Lecture 12]).

Let show that the described above presentation can be computed in polynomial time. If $K = 1$, then the required algorithm is suggested in [26, Theorem 8.7]. If we repeat this algorithm directly for our situation, then it will work in polynomial time if we can solve the following problems:

(AGM) given a set of permutations $g_1, \dots, g_r$ with $r \leq p(n)$ for some polynomial p and a permutation g, test if $gK \in \langle g_1, \dots, g_r, K\rangle/K$;

$(AGMX)$ given an abelian group $H/K = \langle h_1 K\rangle \times \cdots \times \langle h_t K\rangle$ with $t \leq 2n$, order of every $h_i K$ is not greater than n and $hK \in H$, find a_i such that $hK = \prod_{i=1}^{t} h_i^{a_i} K$.

To do (AGM) we need only to test if $g \in \langle g_1, \dots, g_r, K\rangle$. The solution to this problem is well known (for example, see [15, p. 49]). Let $H_i = \langle h_i, K\rangle$. It is clear that $H_i \trianglelefteq H$ for all i. Therefore by [27, Theorem 7.1(6)] we can find in polynomial time $x_i \in H_i$ such that $h = \prod_{i=1}^{t} x_i$ and $t \leq 2n$. Now $x_i K \in \langle h_i K\rangle$ for every i. Hence in polynomial time in n we can find $a_i \leq n$ with $x_i K = h_i^{a_i} K$. Thus $(AGMX)$ can also be solved in polynomial time. It means that the described above presentation of G/K can be computed in polynomial time in n as so do $\Omega_1(G/K)$. □

Lemma 6. *In the notations of Theorem 1 and Lemma 3 if H/T is abelian and Z_1 is given, then we can compute Z_2 in polynomial time in n.*

Proof. Assume that H/T is abelian. If $\mathrm{f}(H/T, G) = 0$, then $Z_1 = Z_2$ by Lemma 3. So we can assume that $\mathrm{f}(H/T, G) = 1$. Note that H/T is a p-group for some prime p and every minimal normal p-subgroup of a group is centralized by its the greatest normal p-subgroup [2, A, Lemma 13.6]. Since every minimal normal p-subgroup is generated by elements of order p, all minimal normal p-subgroups of H/Z_1 are subgroups of $P/Z_1 = \Omega_1(\mathrm{Z}(\mathrm{O}_p(H/Z_1)))$.

If $P \leq T$, then $H/Z_1 \neq T/Z_1 \times Z/Z_1$ for all $Z \trianglelefteq G$. Thus $Z_2 = Z_1$ by Lemma 3. Assume now that $P \not\leq T$. From the Krull–Remak–Schmidt Theorem [2, A, Theorem 4.9] it follows that P/Z_1 is the direct product of G-indecomposable subgroups $P_1/Z_1, \dots, P_m/Z_1$. Note that $H/Z_1 = T/Z_1 \times Z/Z_1$ for some $Z \trianglelefteq G$ iff $P/Z_1 = (P \cap T)/Z_1 \times Z/Z_1$ for some G-simple subgroup Z/Z_1 of P/Z_1. Let $(P \cap T)/Z_1$ be the direct product of G-indecomposable subgroups $T_1/Z_1, \dots, T_l/Z_1$. If the required Z exists, then P/Z_1 is the direct product of G-indecomposable subgroups $T_1/Z_1, \dots, T_l/Z_1$ and Z/Z_1. Hence $l = m-1$. Moreover if P/Z_1 is the direct product of G-indecomposable subgroups $Q_1/Z_1, \dots,$ Q_m/Z_1, then these subgroups can be numbered in such way that P/Z_1 is the direct product of $T_1/Z_1, \dots, T_{m-1}/Z_1$ and Q_m/Z_1 by the Krull–Remak–Schmidt Theorem [2, A, Theorem 4.9]. In this case Q_m/Z_1 is G-simple and not contained in T/Z_1. Therefore the required subgroup Z exists only in case when any direct decomposition of P/Z_1 into the product of G-indecomposable subgroups has G-simple subgroup not contained in T/Z_1.

Note that P/Z_1 can be viewed as a G-module over $\mathbb{F}_p$ and G-indecomposable and G-simple subgroups of $P/Z_1 \leq G/Z_1$ are in one to one correspondence of indecomposable and simple submodules of a G-module P/Z_1.

From $(A4)$ it follows that we can compute $\mathrm{Z}(\mathrm{O}_p(H/Z_1))$ in polynomial time. Since $\mathrm{Z}(\mathrm{O}_p(H/Z_1))$ is abelian, we can find P/Z_1 in polynomial time by Lemma 5. According to [15, p. 155] for each generator g of G in polynomial time we can find the linear transformation which this element induces on P/Z_1. Let R be the algebra generated by the above mentioned transformations. Now by [28, Theorem 5] we can decompose P/Z_1 into the sum $P_1 \oplus \cdots \oplus P_k$ of indecomposable submodules in polynomial time. For each P_i we can check either it is simple or not by [29, Corollary 5.4] and for each simple P_i we can find the subgroup P_i/Z_1 of P/Z to which it corresponds. If $P_i/Z \not\leq T/Z$, then P_i/Z is the required subgroup. □

Proof of Theorem 1

Note that $\mathrm{Z}(G, \mathrm{f})$ is well defined by (1) of Lemma 2. We can compute a chief series $K = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m = G$ of G in polynomial time by $(A5)$. Let $Z_i = G_i \cap \mathrm{Z}(G/K, \mathrm{f})$. We see that $Z_{i-1} = Z_i \cap G_{i-1}$ and $Z_0 = K$. Now using Lemmas 4 and 6 if we know Z_{i-1} then we can compute Z_i in polynomial time. Since $m \leq 2n$ by Lemma 1, we can compute $\mathrm{Z}(G/K, \mathrm{f})$ in polynomial time.

5. Applications

Recall [14] that $\mathrm{Int}_{\mathfrak{F}}(G)$ denotes the intersection of all $\mathfrak{F}$-maximal subgroups of G for a formation $\mathfrak{F}$. Shemetkov posed the following question on Gomel Algebraic seminar in 1995: "For what non-empty (normally) hereditary local (Baer-local) formations $\mathfrak{F}$ do the equality $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ hold for every group G?" The solution to this question in case when $\mathfrak{F}$ is a hereditary saturated formation was obtained in [14]. For some class of Baer-local formations the answer to this question was given in [30]. Nevertheless this problem is still open. We will show that Theorem 1 can be used to compute $\mathrm{Int}_{\mathfrak{F}}(G)$ for some formations $\mathfrak{F}$.

5.1. Baer-local and local formation

The notion of $\mathfrak{F}$-hypercenter plays an important role in the study of Baer-local and local formations. Let f be a function which assigns to every simple group J a possibly empty formation $f(J)$. Now extend the domain of f. If G is the direct product of simple groups isomorphic to J, then let $f(G) = f(J)$. If J is a cyclic group of order p, then let $f(p) = f(J)$. Such functions f are called Baer functions [2, IV, Definitions 4.9]. A formation $\mathfrak{F}$ is called Baer-local (or composition, see [3, p. 4] and [18]) if for some Baer function f holds:

$$\mathfrak{F} = (G \mid G/C_G(H/K) \in f(H/K) \text{ for every chief factor } H/K \text{ of } G).$$

Theorem 2. *Let $\mathfrak{F}$ be a Baer-local formation defined by f. Assume that $(G/K)^{f(J)}$ can be computed in polynomial time for every $K \trianglelefteq G \leq S_n$ and a simple group J. Then $\mathrm{Z}_{\mathfrak{F}}(G/K)$ can be computed in polynomial time for every $K \trianglelefteq G \leq S_n$.*

Proof. Here we assume that if $f(J) = \varnothing$, then the computation of $(G/K)^{f(J)}$ returns "is not defined". Let $\mathrm{f}(H/K, G) = 1$ iff $G/C_G(H/K) \in f(H/K)$. This condition is equivalent to $G^{f(H/K)}$ is defined and $[G^{f(H/K)}, H] \subseteq K$. Hence it can be checked in polynomial time by $(A3)$, $(A6)$ and the statement of the theorem. By analogy with the proof of [9, Theorem 7] one can show that f is a chief factor function. Then $\mathfrak{F} = \mathcal{C}(\mathrm{f})$. Hence we can compute $G^{\mathfrak{F}}$ in polynomial time by [9, Algorithm 4].

Let H/K be a chief factor of G and $T \simeq (H/K) \rtimes G/C_G(H/K)$. Note that T is a primitive group of type 1 or 3 (see [2, A, Theorem 15.2]). Then $T/C_T(H/K) \simeq G/C_G(H/K)$ by [2, A, Theorem 15.2(1, 3)]. So if $T \in \mathfrak{F}$, then $G/C_G(H/K) \in f(H/K) \cap \mathfrak{F}$. From the other hand if $G/C_G(H/K) \in f(H/K) \cap \mathfrak{F}$, then $T \in \mathfrak{F}$. It is clear that

$$G/C_G(H/K) \in (f(H/K) \cap \mathfrak{F})$$

iff $[G^{\mathfrak{F}}G^{f(H/K)}, H] \subseteq K$. Let $\mathrm{f}_1(H/K, G) = 1$ iff $G^{f(H/K)}$ is defined and $[G^{\mathfrak{F}}G^{f(H/K)}, H] \subseteq K$. Note that we can check this condition in polynomial time by $(A3)$ and $(A6)$, f_1 is a chief factor function and $\mathrm{Z}(G, \mathrm{f}_1) = \mathrm{Z}_{\mathfrak{F}}(G)$ for any group G. Thus $\mathrm{Z}_{\mathfrak{F}}(G/K)$ can be computed in polynomial time for any $K \trianglelefteq G \leq S_n$ by Theorem 1. □

A Baer-local formation defined by Baer function f is called local if $f(J) = \cap_{p\in\pi(J)} f(p)$ for every simple group J. In this case to define $\mathfrak{F}$ we need only to know the values of f on primes.

Corollary 1. *Let $\mathfrak{F}$ be a local formation locally defined by f. Assume that $(G/K)^{f(p)}$ can be computed in polynomial time for every $K \trianglelefteq G \leq S_n$ and a prime p. Then $\mathrm{Z}_{\mathfrak{F}}(G/K)$ can be computed in polynomial time for every $K \trianglelefteq G \leq S_n$.*

According to [9, the proof of Corollary 1] for the class $\mathfrak{U}$ of supersoluble groups, the class $w\mathfrak{U}$ of widely supersoluble groups [31], the class $\mathfrak{N}\mathcal{A}$ of groups G such that all Sylow subgroups of $G/\mathrm{F}(G)$ are abelian [31], the class $sm\mathfrak{U}$ of groups with submodular Sylow subgroups [32; 33], the class of strongly supersoluble groups $s\mathfrak{U}$ [32] and the class $sh\mathfrak{U}$ of groups all of whose Schmidt subgroups are supersoluble [34] we can compute the described in the statement of Corollary 1 $f(p)$-residuals in polynomial time.

Corollary 2. *Let $\mathfrak{F} \in \{\mathfrak{U}, \mathrm{w}\mathfrak{U}, s\mathfrak{U}, sm\mathfrak{U}, \mathfrak{N}\mathcal{A}, sh\mathfrak{U}\}$. Then $\mathrm{Z}_{\mathfrak{F}}(G/K)$ can be computed in a polynomial time for every $K \trianglelefteq G \leq S_n$.*

Recall [37] that a group is called *c-supersoluble* in the terminology of Vedernikov (SC-group in the terminology of Robinson [35]) if every its chief factor is a simple group. It was proved the the class $\mathfrak{U}_c$ of all c-supersoluble groups is a Baer-local formation [37].

Theorem 3. *In polynomial time one can compute $\mathrm{Z}_{\mathfrak{U}_c}(G/K)$ for $K \trianglelefteq G \leq S_n$.*

Proof. Let f_1 checks if H/K is simple. Then $\mathrm{f}_1(H/K, G)$ can be computed in polynomial time by $(A7)$. Therefore $\mathfrak{U}_c = \mathcal{C}(\mathrm{f}_1)$. It is easily seen that an abelian chief factor is $\mathfrak{U}_c$-central iff it is simple. If H/K is a non-abelian factor, then $T = (H/K) \rtimes (G/C_G(H/K))$ has two minimal normal subgroups whose quotients are isomorphic to $G/C_G(H/K)$ by [2, A, Theorem 15.2(3)]. Now $T \in \mathfrak{U}_c$ iff $G/C_G(H/K) \in \mathfrak{U}_c$. The last is equivalent to $[G^{\mathfrak{U}_c}, H] \subseteq K$. According to [9, Theorem 1] we can compute $G^{\mathfrak{U}_c}$ in polynomial time. Therefore we can check if H/K is $\mathfrak{U}_c$-central in G in polynomial time and hence compute $\mathrm{Z}_{\mathfrak{U}_c}(G/K)$. □

Recall that a group is called quasinilpotent if every its element induces an inner automorphism on every its chief factor. The class of all quasinilpotent groups is denoted by $\mathfrak{N}^*$.

Theorem 4. *In polynomial time one can compute $\mathrm{Int}_{\mathfrak{N}^*}(G/K) = \mathrm{Z}_{\mathfrak{N}^*}(G/K)$ for $K \trianglelefteq G \leq S_n$.*

Proof. According to [30, Corollary 1] $\mathrm{Int}_{\mathfrak{N}^*}(G) = \mathrm{Z}_{\mathfrak{N}^*}(G)$ holds for every group G. From [30, Remark 2] $\mathrm{Z}_{\mathfrak{N}^*}(G)$ is the greatest normal subgroup of G such that every element of G induces an inner automorphism on every chief factor of G below $\mathrm{Z}_{\mathfrak{N}^*}(G)$, i.e. $\mathrm{Z}_{\mathfrak{N}^*}(G) = \mathrm{Z}(G, \mathrm{f})$ where f checks if every element of G induces an inner automorphism on H/K that is $G = HC_G(H/K)$ or $G/K = (H/K)C_{G/K}(H/K)$. This condition can be checked in polynomial time by $(A1), (A3)$ and $(A6)$. It is straightforward to check that f is a chief factor function. Thus in polynomial time one can compute $\mathrm{Int}_{\mathfrak{N}^*}(G/K) = \mathrm{Z}_{\mathfrak{N}^*}(G/K)$ for $K \trianglelefteq G \leq S_n$ by Theorem 1. □

5.2. Formations with the Shemetkov property

Recall [36] (see also [1, Chapter 6]) that a formation $\mathfrak{F}$ has the Shemetkov property (resp. in the class of all soluble groups $\mathfrak{S}$) if every (resp. soluble) $\mathfrak{F}$-critical group (i.e. non-$\mathfrak{F}$-group all whose proper subgroups belong to $\mathfrak{F}$) is either a Schmidt group (i.e. non-nilpotent group all whose proper subgroups are nilpotent) or a cyclic group of prime order. If a hereditary local formation $\mathfrak{F}$ has the

Shemetkov property (and contains all nilpotent groups) then it can be locally defined by f where $f(p) = \mathfrak{G}_{g(p)}$ and g assigns to a prime p a set of primes $g(p)$ with $p \in g(p)$ [1, Corollary 6.4.5]. The converse is not true. Nevertheless a hereditary local formation $\mathfrak{F}$ of soluble groups has the Shemetkov property in $\mathfrak{S}$ and contains all nilpotent groups iff $\mathfrak{F}$ is locally defined by f where $f(p) = \mathfrak{S}_{g(p)}$ and g assigns to a prime p a set of primes $g(p)$ with $p \in g(p)$ [36]. With the help of the next result one can compute the $\mathfrak{F}$-hypercenter and the intersection of all maximal $\mathfrak{F}$-subgroups of a (resp. soluble) group G where $\mathfrak{F}$ is a hereditary formation with the Shemetkov property (resp. in $\mathfrak{S}$).

Theorem 5. *Let g be a function which assigns to a prime p a set of primes $g(p)$ with $p \in g(p)$ and $h(p) = \mathfrak{G}_{g(p)}$. Assume that $g(p) \cap \pi(G)$ can be computed in a polynomial time for every $p \in \pi(G)$. Let $\mathfrak{F}$ be a local formation defined by h. Then $\mathrm{Int}_{\mathfrak{F}}(G/K)$ and $\mathrm{Z}_{\mathfrak{F}}(G/K)$ can be computed in a polynomial time for every $K \trianglelefteq G \leq S_n$.*

Proof. Let prove that $H/K \leq \mathrm{Int}_{\mathfrak{F}}(G/K)$ iff $G/C_G(H/K) \in h(p)$ for every $p \in \pi(H/K)$.

Assume that $G/C_G(H/K) \in h(p)$ for every $p \in \pi(H/K)$. Let M/K be an $\mathfrak{F}$-maximal subgroup of G/K. Let $T = MH$. Note that $T/C_T(U/V) \in h(p)$ for all $p \in \pi(U/V)$ and every chief factor U/V of T/K below H/K. From $T/H \in \mathfrak{F}$ it follows that $T/C_T(U/V) \in h(p)$ for all $p \in \pi(U/V)$ and every chief factor U/V of T/K above H/K. Thus $T/K \in \mathfrak{F}$. Hence $T/K = M/K$. Therefore $H/K \leq \mathrm{Int}_{\mathfrak{F}}(G/K)$.

Assume that $H/K \leq \mathrm{Int}_{\mathfrak{F}}(G/K)$. It means that $\pi(H/K) \subseteq g(p)$ for all $p \in \pi(H/K)$. Let M/K be an $\mathfrak{F}$-maximal subgroup of G/K and $K = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_m = H$ be a part of chief series of M/K. Then $(M/K)/C_{M/K}(H_i/H_{i-1}) \in h(p)$ for all $p \in \pi(H/K)$. Let $C/K = \cap_{i=1}^m C_{M/K}(H_i/H_{i-1})$. Now $(C/K)/C_{M/K}(H/K)$ is a $\pi(H/K)$-group by [2, A, Corollary 12.4(a)]. Since $h(p)$ is a formation, $(M/K)/(C/K) \in h(p)$ for all $p \in \pi(H/K)$. Thus $(M/K)/C_{M/K}(H/K) \in \mathfrak{G}_{g(p)} = h(p)$ for all $p \in \pi(H/K)$. Since $g(p) \neq \varnothing$ for all prime p, we see that every element of G/K belongs to some its $\mathfrak{F}$-maximal subgroup. From $(M/K)/C_{M/K}(H/K) \simeq MC_{G/K}(H/K)/C_{G/K}(H/K)$ it follows that $G/C_G(H/K)$ is a $\mathfrak{G}_{g(p)}$-group for every $p \in \pi(H/K)$, i.e. $G/C_G(H/K) \in h(p)$ for every $p \in \pi(H/K)$.

From [14, Theorem C(e)] it follows that if $I \leq \mathrm{Int}_{\mathfrak{F}}(G)$, then $\mathrm{Int}_{\mathfrak{F}}(G/I) = \mathrm{Int}_{\mathfrak{F}}(G)/I$. Therefore $\mathrm{Int}_{\mathfrak{F}}(G/K) = \mathrm{Z}(G/K, \mathrm{f})$ where f checks if $G/C_G(H/K) \in h(p)$ for every $p \in \pi(H/K)$. The last condition is equivalent to $[\mathrm{O}^{g(p)}(G), H] \subseteq K$ and can be checked in polynomial time by $(A3)$, $(A4)$ and $(A6)$. Thus the statement of theorem follows from Theorem 1 and Corollary 1. □

Let φ be some linear ordering on $\mathbb{P}$. Recall [2, IV, Examples 3.4(g)] that a group G is called a Sylow tower group of type φ if it has normal Hall $\{p_1, \dots, p_t\}$-subgroups for all $1 \leq t \leq k$ where $\pi(G) = \{p_1, \dots, p_k\}$ and $p_i >_\varphi p_j$ for $i < j$. Let $g(p) = \{q \mid q \leq_\varphi p\}$. Note that the class of all Sylow tower group of type φ can be locally defined by $h(p) = \mathfrak{G}_{g(p)}$.

Corollary 3. *Let φ be a linear ordering on $\mathbb{P}$ such that we can check in polynomial time if $p \leq_\varphi q$ for any primes $p, q \leq n$. If $\mathfrak{F}$ is the class of all Sylow tower group of type φ, then $\mathrm{Int}_{\mathfrak{F}}(G/K)$ and $\mathrm{Z}_{\mathfrak{F}}(G/K)$ can be computed in a polynomial time for every $K \trianglelefteq G \leq S_n$.*

It is well known that the class of all p-nilpotent groups can be locally defined by h where $h(p) = \mathfrak{G}_p$ and $h(q) = \mathfrak{G}$ for $q \neq p$.

Corollary 4. *Let p be a prime. If $\mathfrak{F}$ is the class of all p-nilpotent groups, then $\mathrm{Int}_{\mathfrak{F}}(G/K)$ and $\mathrm{Z}_{\mathfrak{F}}(G/K)$ can be computed in a polynomial time for every $K \trianglelefteq G \leq S_n$.*

5.3. Rank formations

If H/K is a chief factor of G, then $H/K = H_1/K \times \dots \times H_k/K$ where H_i/K are isomorphic simple groups. The number k is called the *rank* of H/K in G.

Definition 3 [2, Chapter VII, Definition 2.3]. A *rank function* R is a map which associates with each prime p a set $R(p)$ of natural numbers. With each rank function R we associate a class of soluble groups

$$\mathfrak{F}(R) = (G \in \mathfrak{S} \mid G \simeq 1 \text{ or for all } p \in \mathbb{P} \text{ each } p\text{-chief factor of } G \text{ has rank in } R(p)).$$

This class is a formation of soluble groups. If $R(p) = \{1, 2\}$ for all $p \in \mathbb{P}$, then formation $\mathfrak{F}(R)$ is non-local by [2, Chapter VII, Theorem 2.18] and hence is non-Baer-local. If $R(p) = \{1\}$ for all $p \in \mathbb{P}$, then $\mathfrak{F}(R) = \mathfrak{U}$.

Theorem 6. *Let R be a rank function. Assume that one can test if $n \in R(p)$ in polynomial (in n) time for every $p \in \mathbb{P}$. Then $\mathrm{Z}_{\mathfrak{F}(R)}(G/K)$ can be computed in polynomial time for every $K \trianglelefteq G \leq S_n$.*

Proof. With the help of $(A3)$ we can compute the rank of H/K in polynomial time. Hence we can check if $G/K \in \mathfrak{F}(\mathcal{R})$ for any $K \trianglelefteq G \leq S_n$ in polynomial time by $(A3)$ and the statement of the theorem. Note that a chief factor H/K is $\mathfrak{F}(R)$-central iff H/K is a p-group for some prime p, the rank of H/K is in $R(p)$ and $G/C_G(H/K) \in \mathfrak{F}(R)$. Since $C_G(H/K)/K = C_{G/K}(H/K)$, we can check if H/K is $\mathfrak{F}(R)$-central in polynomial time by $(A1), (A3), (A5)$. Thus the statement of Theorem 6 follows from Theorem 1.

6. Final Remarks an Open Questions

This work belongs to a series of works devoted to the computational recognition of the formation $\mathfrak{F}$ of finite groups and subgroups associated with $\mathfrak{F}$. The first step [9] was to construct the $\mathfrak{F}$-residual. In this paper we constructed the $\mathfrak{F}$-hypercenter. In the core of these two works was a definition of chief factor function. The $\mathfrak{F}$-residual and the $\mathfrak{F}$-hypercenter are analogues to the final terms of the lower and the upper central series for a formation $\mathfrak{F}$ respectively. The ideas of the construction of these objects were different. This is well illustrated by the fact that $\mathrm{Z}(G, \mathrm{f})$, and hence the computation of the $\mathfrak{F}$-hypercenter, depends on the choice of f, when the formation defined by f is fixed, but the computation of the $\mathfrak{F}$-residual does not depend on f.

For a formation $\mathfrak{F} = \mathcal{C}(\mathrm{f}_0)$ it is natural to ask can one find a good from the computational point of view chief factor function that defines it.

Question 1. Given formation $\mathfrak{F} = \mathcal{C}(\mathrm{f}_0)$ is it possible to construct a function f with $\mathfrak{F} = \mathcal{C}(\mathrm{f})$ such that $\mathrm{f}(H/K, G)$ can be computed in a polynomial time for every permutation group G of degree n and its chief factor H/K. In particular, do there exists formations $\mathfrak{F} = \mathcal{C}(\mathrm{f})$ such that there is no polynomial time algorithms to check if G/K belongs to $\mathfrak{F}$ or not?

Looking through known algorithms for permutation groups [15] one can note that there is no polynomial time algorithm for finding the exponent.

Question 2. Given a permutation group G of degree n and its normal subgroup K can one find in polynomial time the exponent of G/K?

Corollary 1 and [9, Corollary 1] were dedicated to the formations of supersoluble type. In [38] a formation $v\mathfrak{U}$ of groups with cyclic primary $\mathbb{P}$-subnormal subgroups was introduced. This formation is local and its local definition involves formation of groups of exponent dividing m.

Question 3. Given a permutation group G of degree n and its normal subgroup K can one in polynomial time test if $G/K \in v\mathfrak{U}$, compute the $v\mathfrak{U}$-residual and the $v\mathfrak{U}$-hypercenter of G/K?

The constructed in the paper algorithm is purely theoretical. It is reduced to some well known algorithms about permutation groups. Not all of these algorithms were implemented in computer algebra systems. So the next step in this research is to implement Algorithm 1.

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