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AN OPTIMAL INTERPOLATION PROBLEM WITH HERMITE INFORMATION IN THE SOBOLEV CLASS $W_1^n([-1,1])$ [1]

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In this paper, we study the optimal interpolation problem in the Sobolev class $W_1^n([-1,1])$, $n \in \mathbb{N}$, with Hermite information. By some properties of spline functions, we proved that the Lagrange interpolation based on the extreme points of Chebyshev polynomials is optimal for $W_1^n([-1,1])$, and we obtained the approximation error for the optimal interpolation problem.

Keywords: Hermite interpolation, spline function, optimal interpolation, Sobolev class.

Даньдань Го, Йонг Пинг Ли, Гуйцяо Сюй. Оптимальная интерполяционная задача с эрмитовой информацией в классе Соболева $W_1^n([-1,1])$.

В данной работе изучается задача оптимальной интерполяции в классе Соболева $W_1^n([-1,1])$, $n \in \mathbb{N}$ с информацией Эрмита. С помощью некоторых свойств сплайн-функций мы доказали, что интерполяция Лагранжа по экстремальным точкам полиномов Чебышева является оптимальной для $W_1^n([-1,1])$, и получили ошибку аппроксимации для задачи оптимальной интерполяции.

Ключевые слова: эрмитова интерполяция, сплайн-функция, оптимальная интерполяция, класс Соболева.

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1. Introduction and definitions

The study of the best approximation problem is a very important research direction in approximation theory. As an important branch of mathematics, the theory of function approximation should be traced back to the famous theorem that continuous functions can be approximated by polynomials established by Weierstrass in 1885 and the characteristic theorem of the best approximation proposed by Chebyshev in 1859. In the 1950s, on the basis of global approximation, people used interpolation polynomials, rational functions, and so on as approximation tools to carry out in-depth research. The spline function approximation started in the late 1970s and the algebraic polynomial and trigonometric polynomial approximation developed in the 1980s are the development process of approximation theory.

1.1. Notations and definitions

This paper mainly focuses on the optimal interpolation problem with Hermite information in the Sobolev class. As the preparatory work, we first introduce some basic notations and definitions that are of great use for our study.

Denote by $\mathbb{N}$ the set of all positive integers, and let $n, r \in \mathbb{N}$. Let G be a Banach space of functions defined on the compact set $[-1,1]$, and F be the subspace of G that can be continuously embedded in $C^{n-1}([-1,1])$, where $C^n = C^n([-1,1])$ represents the space of functions with nth order continuous derivative on $[-1,1]$, $\mathfrak{M}$ be a convex and central symmetric subset of F, and F_n is an n-dimensional subspace of G. Specially, $C^0([-1,1]) = C([-1,1])$. Following Traub and

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Woźniakowsi [1], we study the problem of optimal recovery for f from $\mathfrak{M}$ by using a finite number of Hermite data $f^{(j)}(t)$ for some $j \leq n-1$ and some $t \in [-1,1]$. We consider only nonadaptive information. Now, we list some basic concepts as follows.

Let $\mathscr{A}_n$ be the class of all sets

$$\Theta := ((x_1,\alpha_1),(x_2,\alpha_2),\ldots,(x_r,\alpha_r))$$

of ordered pairs (x_i,α_i), $i=1,2,\ldots,r$, of Hermite interpolation nodes $-1 \leq x_1 < x_2 < \cdots < x_r \leq 1$ and multiplicity labels $\{\alpha_i \in \mathbb{N} : i=1,2,\ldots,r\}$ with $\sum_{i=1}^r \alpha_i = n$, and call the number of interpolation nodes of a set Θ is n (counting multiplicity).

For each $\Theta = ((x_1,\alpha_1),(x_2,\alpha_2),\ldots,(x_r,\alpha_r)) \in \mathscr{A}_n$, we can determine a mapping $I_\Theta : \mathfrak{M} \longrightarrow \mathbb{R}^n$, and

$$I_\Theta(f) := (f^{(0)}(x_1),\ldots,f^{(\alpha_1-1)}(x_1), f^{(0)}(x_2),\ldots,f^{(\alpha_2-1)}(x_2),\ldots,f^{(0)}(x_r),\ldots,f^{(\alpha_r-1)}(x_r)),$$

where $f^{(0)} := f$.

Usually, a mapping $\varphi : \mathbb{R}^n \longrightarrow F_n$ is called an algorithm. If this mapping is linear, we call it a linear algorithm. Let Φ_n (Φ_n^L) denote the set of all algorithms (linear algorithms).

For any $\varphi \in \Phi_n^L$, it has the following form

$$\varphi \circ I_\Theta(f) := \sum_{i=1}^{r}\sum_{j=0}^{\alpha_i-1} f^{(j)}(x_i)h_{i,j}(x), \quad h_{i,j} \in F_n. \tag{1.1}$$

For any $f \in \mathfrak{M}, \varphi \in \Phi_n, \Theta \in \mathscr{A}_n$, if we take $\varphi \circ I_\Theta(f)$ as an approximation representation of f, then the global error is

$$\sup_{f\in\mathfrak{M}} \| f - \varphi \circ I_\Theta(f) \|_G . \tag{1.2}$$

According to Sun and Fang [2], we get that the intrinsic error and the linear intrinsic error of the optimal recovery problem $(\mathfrak{M}, I_\Theta, G)$ are defined as follows

$$\mathscr{E}(\mathfrak{M},\Theta,G) := \inf_{\varphi\in\Phi_n}\sup_{f\in\mathfrak{M}} \| f - \varphi \circ I_\Theta(f) \|_G,$$

and

$$\mathscr{E}^L(\mathfrak{M},\Theta,G) := \inf_{\varphi\in\Phi_n^L}\sup_{f\in\mathfrak{M}} \| f - \varphi \circ I_\Theta(f) \|_G,$$

respectively. The infima of the intrinsic error and the linear intrinsic error with respect to $\mathscr{A}_n$ are

$$R(n,\mathfrak{M},G) := \inf_{\Theta\in\mathscr{A}_n} \mathscr{E}(\mathfrak{M},\Theta,G) = \inf_{\varphi\in\Phi_n,\,\Theta\in\mathscr{A}_n}\sup_{f\in\mathfrak{M}} \| f - \varphi \circ I_\Theta(f) \|_G, \tag{1.3}$$

and

$$R^L(n,\mathfrak{M},G) := \inf_{\Theta\in\mathscr{A}_n} \mathscr{E}^L(\mathfrak{M},\Theta,G) = \inf_{\varphi\in\Phi_n^L,\,\Theta\in\mathscr{A}_n}\sup_{f\in\mathfrak{M}} \| f - \varphi \circ I_\Theta(f) \|_G,$$

respectively. Obviously, $R(n,\mathfrak{M},G) \leq R^L(n,\mathfrak{M},G)$. For the convenience of the readers, when we only choose the values of the function f in I_Θ, that is, $\alpha_i = 1$, $i=1,2,\ldots,n$, we can replace $R(n,\mathfrak{M},G)$ and $R^L(n,\mathfrak{M},G)$ with $\bar{R}(n,\mathfrak{M},G)$ and $\bar{R}^L(n,\mathfrak{M},G)$, respectively.

If there is $\Theta^* \in \mathscr{A}_n$ and $\varphi^* \in \Phi_n$ that satisfies

$$R(n,\mathfrak{M},G) = \mathscr{E}(\mathfrak{M},\Theta^*,G) = \sup_{f\in\mathfrak{M}} \| f - \varphi^* \circ I_{\Theta^*}(f) \|_G,$$

then we call I_{Θ^*} the nth optimal information operator and φ^* the nth optimal algorithm.

We call

$$D(n,\mathfrak{M},G) := 2\inf_{\Theta\in\mathscr{A}_n}\sup_{f\in\mathfrak{M},\,I_\Theta(f)=0} \| f \|_G \tag{1.4}$$

the n-minimal information diameter. From [1], we can obtain

$$\sup_{f\in\mathfrak{M},\, I_\Theta(f)=0} \| f \|_G \le \mathscr{E}(\mathfrak{M},\Theta,G) \le 2 \sup_{f\in\mathfrak{M},\, I_\Theta(f)=0} \| f \|_G .$$

Then, combine (1.3) with (1.4) to find the lower bound of each term in the above inequalities when Θ are taken over the set $\mathscr{A}_n$, we obtain

$$\frac{1}{2}D(n,\mathfrak{M},G) \le R(n,\mathfrak{M},G) \le D(n,\mathfrak{M},G). \tag{1.5}$$

Sometimes, we can also use the notation of function approximation to express the above quantities. Let $\mathscr{L}_n$ be a set of linear operators A: $\mathfrak{M} \longrightarrow F_n$. For any $A \in \mathscr{L}_n$, we define

$$E(\mathfrak{M},A,F_n)_G := \sup_{f\in\mathfrak{M}} \|f - A(f)\|_G,$$

and call

$$\mathscr{E}_n^L(\mathfrak{M},F_n)_G := \inf_{A\in\mathscr{L}_n} E(\mathfrak{M},A,F_n)_G$$

the best linear approximation of the function class $\mathfrak{M}$ at G by F_n.

Let $I = [a,b] \subset \mathbb{R}$. For $1 \le p \le \infty$, denote by $L_p(I)$ the space of integrable functions defined on I and equipped with the following finite norm

$$\|f\|_{p,I} := \begin{cases} \left(\int_I |f(x)|^p dx\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in I} |f(x)|, & p = \infty. \end{cases}$$

Denote by $L_p^n(I)$ the class of all functions f such that $f^{(n-1)}$ are absolutely continuous on I and $f^{(n)} \in L_p(I)$, set

$$W_p^n(I) := \{f \in L_p^n(I) : \|f^{(n)}\|_{p,I} \le 1\}.$$

Specially, let $L_p^0(I) = L_p(I)$. When $[a,b] = [-1,1]$, we simply write these notations $\|\cdot\|_{p,I}$, $L_p^n(I)$, $L_p(I)$, $W_p^n(I)$ into $\|\cdot\|_p$, L_p^n, L_p, W_p^n, respectively.

Recently, the study of sampling numbers has attracted much interest, and a great number of interesting results [3–6] have been obtained. In addition, derivative values have been used in calculation and design (see [7–9]). To consider the influence of using derivative values on calculation accuracy, in this paper, we will use the properties of spline functions to study the optimal Hermite interpolation of the Sobolev class W_1^n in L_1. Obviously, W_1^n is a center symmetric convex subset of L_1. To show our results, we introduce the Hermite interpolation operator.

Let $\Theta = ((x_1,\alpha_1),(x_2,\alpha_2),\dots,(x_r,\alpha_r)) \in \mathscr{A}_n$. For $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n. Taking the set of functions $h_{i,j} \in \mathcal{P}_{n-1}$ in (1.1) that satisfies the following conditions:

$$h_{i,j}^{(m)}(x_k) = \begin{cases} 1, & k = i,\ m = j, \\ 0, & \text{otherwise}, \end{cases} \tag{1.6}$$

where $1 \le k, i \le r$, $0 \le m, j \le \alpha_i - 1$. We determine an algorithm φ_Θ such that $\varphi_\Theta \circ I_\Theta : W_1^n \longrightarrow \mathcal{P}_{n-1}$ in (1.1) is a Hermite interpolation polynomial operator, which is denoted by $H_\Theta = \varphi_\Theta \circ I_\Theta$, i.e., the Hermite interpolation polynomial $H_\Theta(f)$ of a function $f \in W_1^n$ based on Θ is defined as

$$H_\Theta(f) \in \mathcal{P}_{n-1}, \quad H_\Theta^{(j)}(f,x_i) = f^{(j)}(x_i), \quad 0 \le j \le \alpha_i - 1, \quad 1 \le i \le r. \tag{1.7}$$

In [10], the classical Hermite interpolation formula gives

$$H_\Theta(f,x) = \sum_{i=1}^{r} \frac{W_\Theta(x)}{(x-x_i)^{\alpha_i}} \sum_{j=0}^{\alpha_i-1} f^{(j)}(x_i) \frac{(x-x_i)^j}{j!} \left\{ \frac{(x-x_i)^{\alpha_i}}{W_\Theta(x)} \right\}_{(x_i)}^{(\alpha_i-j-1)},$$

where $\left\{\dfrac{(x-x_i)^{\alpha_i}}{W_\Theta(x)}\right\}_{(x_i)}^{(\alpha_i-j-1)}$ is the function $\dfrac{(x-x_i)^{\alpha_i}}{W_\Theta(x)}$ expanded at x_i to the first $\alpha_i - j$ terms of the Taylor series, and $W_\Theta(x) = \prod_{i=1}^{r}(x-x_i)^{\alpha_i}$. By (1.6), one obtains an explicit formula

$$H_\Theta(f,x) = \sum_{i=1}^{r}\sum_{j=0}^{\alpha_i-1} f^{(j)}(x_i)h_{i,j}(x). \tag{1.8}$$

In particular, if $x_1, x_2, \ldots, x_n$ are n distinct points in $[-1,1]$, i.e., $\Theta := \{-1 \le x_1 < x_2 < \cdots < x_n \le 1,\ \alpha_i = 1,\ i = 1,2,\ldots,n\}$, we obtain the classical Lagrange interpolation formula

$$L_\Theta(f,x) = \sum_{i=1}^{n} f(x_i)\ell_i(x),$$

where

$$\ell_i(x) = \frac{W_\Theta(x)}{(x-x_i)W'_\Theta(x_i)},\ W_\Theta(x) = \prod_{i=1}^{n}(x-x_i).$$

When constructing interpolation algorithms, the selection of interpolation nodes $\Theta \in \mathscr{A}_n$ is very important. In terms of Lagrange interpolation, given a sufficiently smooth function, if a sequence of interpolation nodes is not suitably chosen, then the sequence of interpolation polynomials does not converge to the function as the number of interpolation nodes tends to infinity. A well-known example is Runge's phenomenon. Hence the study of optimal interpolation nodes is a hot research topic (see [11–16] and the references therein). In this paper, we study the optimal interpolation nodes of general Hermite interpolation and give the nth optimal Hermite interpolation nodes when the number of interpolation nodes is fixed at n.

Hermite interpolation is a kind of interpolation that is wider than Lagrange interpolation. It uses not only the function values information but also the derivative values information. Under the condition of using the same amount of information, can increasing the use of derivative values information make the calculation result more accurate? In general the answer is no. In the following, we give the optimal Hermite interpolation nodes to show it.

Let $G = L_1$, $\mathfrak{M} = W_1^n$, $F_n = \mathcal{P}_{n-1}$, $\mathscr{L}_n := \{H_\Theta : \Theta \in \mathscr{A}_n\}$. We have

$$E(W_1^n, H_\Theta, \mathcal{P}_{n-1})_1 = \sup_{f\in W_1^n} \|f - H_\Theta(f)\|_1, \tag{1.9}$$

and

$$\mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1 = \inf_{\Theta\in\mathscr{A}_n} E(W_1^n, H_\Theta, \mathcal{P}_{n-1})_1. \tag{1.10}$$

In particular, if $\alpha_i = 1$, $i = 1,2,\ldots,n$, we substitute $\bar{\mathscr{E}}_n^L(W_1^n, \mathcal{P}_{n-1})_1$ for $\mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1$.

If $\Theta_c \in \mathscr{A}_n$ satisfies

$$E(W_1^n, H_{\Theta_c}, \mathcal{P}_{n-1})_1 = \mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1,$$

we call Θ_c the nth optimal Hermite interpolation nodes and H_{Θ_c} the nth optimal Hermite interpolation operator for W_1^n in L_1. The value $E(W_1^n, H_{\Theta_c}, \mathcal{P}_{n-1})_1$ is called the nth optimal Hermite interpolation error for W_1^n in L_1.

1.2. Introduction

There are many results about the best approximation on the Sobolev class $W_p^r([a,b])$, where $1 \le p \le \infty$. For example, when $[a,b] = [-1,1]$, in 1983, Kofanov [17] studied the best approximation in L_1 for the class W_1^r by $\mathcal{P}_n$. Set

$$E_n(W_1^r, \mathcal{P}_n)_1 := \sup_{f\in W_1^r}\inf_{g\in\mathcal{P}_n} \|f-g\|_1.$$

The main result of the paper [17] reads as follows:

$$E_n(W_1^r,\mathcal{P}_n)_1=\max\left\{\frac{1}{r!}\left|(x+1)^r+2\sum_{i=1}^{n+1}(-1)^i(x-x_i)_+^r\right|:\ x\in[-1,1]\right\},\quad r\ge 2,\quad n\ge r-1,$$

where $x_i=-\cos\dfrac{\pi i}{n+2}$. The above result was announced in [18] for $r=1,2$ and $n\ge r-1$.

Let $[a,b]=[0,1]$. For $1\le p\le\infty$, $n\ge r\ge 2$. Let S_{r-1}^n be the n-dimensional subspace of polynomial splines of degree $r-1$. Shevaldina [19] obtained the exact value of $E_n(W_p^r([0,1]),S_{r-1}^n)_1$, where

$$E_n(W_p^r([0,1]),S_{r-1}^n)_1:=\sup_{f\in W_p^r([0,1])}\inf_{s\in S_{r-1}^n}||f-s||_1.$$

In addition, she determined the spline function of the best approximation in the mean for the class $W_p^r([0,1])$ as follows:

$$E_n(W_p^r([0,1]),S_{r-1}^n)_1=\sup_{f\in W_p^r([0,1])}||f-s(f;\cdot)||_1,$$

where $s(f;x)$ is a spline function interpolating f on knots $x_k=1/2(1-\cos k\pi/(n+1))$, $k=1,\dots,n$, i.e., $s(f;x)=\sum_{j=1}^n f(x_j)s_j(x)$, with the interpolation basis splines $s_j(x)$: $s_j(x_k)=\delta_{j,k}$, ($\delta_{j,k}$—kronecker delta) for $x\in[0,1]$.

Later, in [20], Xu, Liu, and Wang studied the optimal Lagrange interpolation problem in L_1 for the class W_1^r by $\mathcal{P}_{r-1}$, and established a result as follows:

$$\bar{R}(r,W_1^r,L_1)=\bar{R}^L(r,W_1^r,L_1)=\bar{\mathscr{E}}_r^L(W_1^r,\mathcal{P}_{r-1})_1=E(W_1^r,L_{\Theta_r},\mathcal{P}_{r-1})_1=\frac{C_r}{r!},\quad r\in\mathbb{N},$$

where

$$\Theta_r=\Big(\cos\frac{r\pi}{r+1},\cos\frac{(r-1)\pi}{r+1},\dots,\cos\frac{\pi}{r+1}\Big)$$

is the set of extreme points of $(r+1)$th Chebyshev polynomial $T_{r+1}(x)=\cos((r+1)\arccos x)$ and

$$C_r=\left\|(1-\cdot)^r-2\sum_{i=1}^r(-1)^{i-1}\Big(\cos\frac{i\pi}{r+1}-\cdot\Big)_+^r\right\|_\infty.$$

In the present paper, we study the optimal Hermite interpolation problem for the class W_1^n by $\mathcal{P}_{n-1}$. The problem is to find the optimal Hermite interpolation process. From the above analysis, the result of the paper [17] (for $n=r-1$) gives the lower estimate of the approximation error for this problem. In addition, the result in [19] for case $n=r$ and when the space of spline functions coincides with the space of algebraic polynomials provides the corresponding above estimate. Next, we will present a novel and significantly different solution to this problem.

2. Main results and preliminaries

Theorem 2.1. *Let $n\in\mathbb{N}$. Then we have*

$$R(n,W_1^n,L_1)=R^L(n,W_1^n,L_1)=\mathscr{E}_n^L(W_1^n,\mathcal{P}_{n-1})_1=E(W_1^n,L_{\Theta_n},\mathcal{P}_{n-1})_1,$$

where

$$\Theta_n=\Big(\cos\frac{n\pi}{n+1},\cos\frac{(n-1)\pi}{n+1},\dots,\cos\frac{\pi}{n+1}\Big)$$

is the set of extreme points of $(n+1)$th Chebyshev polynomial $T_{n+1}(x)=\cos((n+1)\arccos x)$.

Theorem 2.1 shows that the optimal Hermite interpolation is the Lagrange interpolation over the nodes Θ_n, i.e., increasing the use of the derivative values information does not obtain better algorithms. Then, the result of [20] is applied, which contains the approximation error of the optimal Lagrange interpolation.

Now, we introduce some information about the norms of integral operators. Let $K(x,t)$ be a piecewise continuous function on $[-1,1]\times[-1,1]$. We define

$$S(f,x)=\int\limits_{-1}^{1}K(x,t)f(t)dt.$$

It is known that S is a linear continuous operator from L_1 to L_1. Furthermore, let $\|S\|_{1,1}$ be the operator norm of S from L_1 to L_1. Then it is known that

$$\|S\|_{1,1}=\sup_{f\in L_1,f\neq 0}\frac{\|Sf\|_1}{\|f\|_1}=\sup_{-1\le t\le 1}\int\limits_{-1}^{1}|K(x,t)|dx. \tag{2.1}$$

We shall use the following lemmas and properties in the proof of Theorem 2.1. In the first lemma, we give an expression of the remainder of the Hermite interpolation on the interval $[-1,1]$.

Lemma 2.1 *Let $f\in W_1^n$, $\Theta\in\mathscr{A}_n$. Then, the remainder $R_\Theta(f,x):=f(x)-H_\Theta(f,x)$ for the Hermite interpolation polynomial based on Θ can be represented in the form*

$$R_\Theta(f,x)=\frac{1}{(n-1)!}\int\limits_{-1}^{1}f^{(n)}(t)[(x-t)_+^{n-1}-H_\Theta((\cdot-t)_+^{n-1},x)]dt.$$

Proof. Let $P(x)=f(-1)+f'(-1)(x+1)+\cdots+\dfrac{f^{(n-1)}(-1)}{(n-1)!}(x+1)^{n-1}$. For $f\in W_1^n$, we apply Taylor's formula of f at -1 with integral remainder, then $f(x)=P(x)+r_{n-1}(x)$, where the integral remainder r_{n-1} is

$$r_{n-1}(x)=\frac{1}{(n-1)!}\int\limits_{-1}^{1}(x-t)_+^{n-1}f^{(n)}(t)dt.$$

So we can obtain

$$r_{n-1}^{(j)}(x)=\frac{1}{(n-j-1)!}\int\limits_{-1}^{1}(x-t)_+^{n-1-j}f^{(n)}(t)dt,\quad j=0,1,2,\dots,n-1.$$

Considering $P(x)$ is an algebraic polynomial of degree at most $n-1$, and according to the properties of H_Θ, we get $P(x)-H_\Theta(P,x)=0$, where $\Theta\in\mathscr{A}_n$. Hence, by (1.8) we have

$$f(x)-H_\Theta(f,x)=r_{n-1}(x)-H_\Theta(r_{n-1},x)$$

$$=\frac{1}{(n-1)!}\int\limits_{-1}^{1}f^{(n)}(t)\left[(x-t)_+^{n-1}-\sum_{i=1}^{r}\sum_{j=0}^{\alpha_i-1}\frac{(n-1)!}{(n-j-1)!}(x_i-t)_+^{n-1-j}\cdot h_{i,j}(x)\right]dt$$

$$=\frac{1}{(n-1)!}\int\limits_{-1}^{1}f^{(n)}(t)\left[(x-t)_+^{n-1}-H_\Theta\left((\cdot-t)_+^{n-1},x\right)\right]dt.$$

We complete the proof of Lemma 2.1.

Using Lemma 2.1, we get the following lemma.

Lemma 2.2. *Let $\Theta \in \mathscr{A}_n$. Then we have*

$$E(W_1^n, H_\Theta, \mathcal{P}_{n-1})_1 = \sup_{-1\le t\le 1} \int_{-1}^{1} |B_\Theta(x,t)|dx,$$

where

$$B_\Theta(x,t) = \frac{(x-t)_+^{n-1} - H_\Theta\left((\cdot - t)_+^{n-1}, x\right)}{(n-1)!}.$$

Proof. Let $B_\Theta(x,t) = \dfrac{(x-t)_+^{n-1} - H_\Theta\left((\cdot - t)_+^{n-1}, x\right)}{(n-1)!}$. For $f \in W_1^n$, then it follows from Lemma 2.1 that

$$f(x) - H_\Theta(f,x) = \int_{-1}^{1} B_\Theta(x,t) f^{(n)}(t)dt.$$

Let

$$S(f^{(n)}, x) = \int_{-1}^{1} B_\Theta(x,t) f^{(n)}(t)dt.$$

Combining (1.9) with (2.1), we obtain

$$E(W_1^n, H_\Theta, \mathcal{P}_{n-1})_1 = \sup_{f\in W_1^n} \|S(f^{(n)},\cdot)\|_1 = \|S\|_{1,1} = \sup_{-1\le t\le 1} \int_{-1}^{1} |B_\Theta(x,t)|dx.$$

We complete the proof of Lemma 2.2.

In the following, we introduce weak Chebyshev subspace and give some properties on this space.

An n-dimensional subspace G of $C([a,b])$ is called a weak Chebyshev subspace if every function $g \in G$ has at most $n-1$ sign changes.

Let points $a = x_0 < x_1 < \cdots < x_r < x_{r+1} = b$ and an integer $m \ge 1$ be given. We call

$$S_m(x_1, x_2, \ldots, x_r) = \{s \in C^{m-1}([a,b]) : \ s\big|_{[x_i, x_{i+1}]} \in \mathcal{P}_m,\ i = 0, 1, \ldots, r\}$$

the space of polynomial splines of degree m with r fixed knots $x_1, x_2, \ldots, x_r$.

The following lemma says that spline spaces are weak Chebyshev subspaces.

Lemma 2.3 [21, Theorem 1.19 of Chapter II]. *The space $S_m(x_1, x_2, \ldots, x_r)$ is a $(r+m+1)$-dimensional weak Chebyshev subspace of $C([a,b])$.*

From [21, Theorem 6.3 of Chapter II], it follows that for every n-dimensional weak Chebyshev subspace of $C([a,b])$, there exists a set of n-canonical points $t_1 < \cdots < t_n$ in (a,b), i.e., there exist $t_1 < \cdots < t_n$ in (a,b) such that

$$\sum_{i=0}^{n} (-1)^i \int_{t_i}^{t_{i+1}} g(t)dt = 0$$

holds for all $g \in G$, where $t_0 = a$ and $t_{n+1} = b$.

If G is a weak Chebyshev subspace of $C([a,b])$, then the set

$$K(G) = \{f \in C([a,b]) : \operatorname{span}(G \cup f) \text{ is a weak Chebyshev subspace of } C([a,b])\}$$

is called the convexity cone of G.

The following lemma gives a special relationship between the best L_1-approximation and interpolation for the weak Chebyshev subspace.

Lemma 2.4 [21, Theorem 6.6 of Chapter II]. *Let G be an n-dimensional weak Chebyshev subspace of $C([a,b])$. If the set $\{t_1,\ldots,t_n\}$ of canonical points of G is poised with respect to G, then every function $f \in K(G)$ has a unique best L_1-approximation g_f from G and g_f is uniquely determined by*

$$g_f(t_i) = f(t_i), \quad i = 1,\ldots,n.$$

3. Proof of Theorem 2.1

The first step is to prove that

$$R(n, W_1^n, L_1) = R^L(n, W_1^n, L_1) = \mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1. \tag{3.1}$$

We first consider the above estimate of (3.1). Let $\Theta \in \mathscr{A}_n$. Then, the Hermite interpolation operator H_Θ is linear. Hence, it follows from (1.10) that

$$R(n, W_1^n, L_1) \le R^L(n, W_1^n, L_1) \le \mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1. \tag{3.2}$$

Now, we consider the lower estimate of (3.1). For $\Theta \in \mathscr{A}_n$, and any $f \in W_1^n$. Let $\bar f = f - H_\Theta(f)$, we can get $\bar f \in W_1^n$. And from (1.7) we know that $\bar f^{(j)}(x_i) = 0,\ 0 \le j \le \alpha_i - 1,\ 1 \le i \le r$. Then

$$\sup_{f\in W_1^n,\ I_\Theta(f)=0} \|f\|_1 \ge \|\bar f\|_1 = \|f - H_\Theta(f)\|_1.$$

Because f is arbitrary, we have

$$\sup_{f\in W_1^n,\ I_\Theta(f)=0} \|f\|_1 \ge \sup_{f\in W_1^n} \|f - H_\Theta(f)\|_1.$$

Therefore, by (1.4) and (1.10), we obtain

$$\frac{1}{2} D(n, W_1^n, L_1) \ge \mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1.$$

Additionally, from (1.5), we know that $R(n, W_1^n, L_1) \ge \frac{1}{2} D(n, W_1^n, L_1)$, so

$$R(n, W_1^n, L_1) \ge \mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1. \tag{3.3}$$

Combining (3.2) with (3.3), we get (3.1).

The next step is to prove that

$$\mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1 = E(W_1^n, L_{\Theta_n}, \mathcal{P}_{n-1})_1. \tag{3.4}$$

Considering L_{Θ_n} is a Hermite interpolation operator, so

$$\mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1 \le E(W_1^n, L_{\Theta_n}, \mathcal{P}_{n-1})_1. \tag{3.5}$$

Finally, we prove the opposite inequality of (3.5). For $t = 1$, $(\cdot - t)_+^{n-1} = 0 \in \mathcal{P}_{n-1}$. For $t = -1$, $(\cdot - t)_+^{n-1} = (\cdot - t)^{n-1} \in \mathcal{P}_{n-1}$. Hence

$$(x-t)_+^{n-1} - L_{\Theta_n}((\cdot - t)_+^{n-1}, x) = 0, \quad (x-t)_+^{n-1} - H_\Theta((\cdot - t)_+^{n-1}, x) = 0. \tag{3.6}$$

Now, we consider $t \in (-1, 1)$. It is known that $\mathcal{P}_n$ is a Chebyshev subspace of $C([-1, 1])$. Furthermore, from [21, Theorem 4.10 of Chapter I], we know that the canonical points for $\mathcal{P}_n$ on $[-1, 1]$ are the extreme points of the Chebyshev polynomial T_{n+2} in $(-1, 1)$, i.e.,

$$t_i = \cos \frac{(n+2-i)\pi}{n+2}, \quad i = 1, 2, \ldots, n+1.$$

So the set Θ_n of the canonical points for $\mathcal{P}_{n-1}$ on $[-1, 1]$ can be expressed as

$$\Theta_n = \left(\cos \frac{n\pi}{n+1}, \cos \frac{(n-1)\pi}{n+1}, \ldots, \cos \frac{\pi}{n+1} \right).$$

In addition, from Lemma 2.3, we know that for each $t \in (-1, 1)$, $\operatorname{span}(\mathcal{P}_{n-1} \cup (\cdot - t)_+^{n-1})$ is a $(n+1)$-dimensional weak Chebyshev space of $C([-1, 1])$, and by the definition of convexity cone of $\mathcal{P}_{n-1}$ we get $(\cdot - t)_+^{n-1} \in K(\mathcal{P}_{n-1})$. Hence, from Lemma 2.4, it follows that $L_{\Theta_n}((\cdot - t)_+^{n-1}, x)$ is the best L_1-approximation of $(\cdot - t)_+^{n-1}$ from $\mathcal{P}_{n-1}$ on $[-1, 1]$. And because $H_\Theta((\cdot - t)_+^{n-1}, x) \in \mathcal{P}_{n-1}$, we get

$$\int_{-1}^{1} \left| (x-t)_+^{n-1} - L_{\Theta_n}((\cdot - t)_+^{n-1}, x) \right| dx \le \int_{-1}^{1} \left| (x-t)_+^{n-1} - H_\Theta((\cdot - t)_+^{n-1}, x) \right| dx. \tag{3.7}$$

From (3.6), (3.7) and Lemma 2.2, it follows that

$$E(W_1^n, L_{\Theta_n}, \mathcal{P}_{n-1})_1 \le E(W_1^n, H_\Theta, \mathcal{P}_{n-1})_1.$$

By (1.10) and the fact that $\Theta \in \mathscr{A}_n$ is arbitrary, hence

$$E(W_1^n, L_{\Theta_n}, \mathcal{P}_{n-1})_1 \le \mathscr{E}_n^L(W_1^n, \mathcal{P}_{n-1})_1.$$

This shows that (3.4) holds.

Sum up, by (3.1) and (3.4), we complete the proof of Theorem 2.1. □

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