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2023 URAL WORKSHOP ON GROUP THEORY AND COMBINATORICS

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A review of the main events of the 2023 Ural Workshop on Group Theory and Combinatorics, held online during the period 21 to 27 August 2023, is presented, and a list of open problems with comments is given. Open problems were formulated by the participants at the Open Problems Session held on August 27, 2023.

Keywords: power graph, enhanced power graph, independence graph of a group, rank graph of a group, finite group, isomorphism of groups, π-solvable group, simple group, average element order, solvable group, deficient element, locally finite group, distance-regular graph, Krein graph, strongly regular graph, Gruenberg–Kegel graph (prime graph), almost simple group, Cayley graph, clique graph, edge-transitive graph, normal cover of a graph, 2-arc-transitive graph, semisymmetric graph, complete class of groups, Baer–Suzuki width, symmetric boundary of a class of groups.

Н. В. Маслова. Международная конференция “2023 Ural Workshop on Group Theory and Combinatorics”.

В статье представлен обзор основных событий Международной конференции “2023 Ural Workshop on Group Theory and Combinatorics”, которая прошла в онлайн формате 21–27 августа 2023 г. Также в статье представлен список открытых проблем, сформулированных участниками на Часе открытых проблем, прошедшем 27 августа 2023 г., и комментарии к этим проблемам.

Ключевые слова: граф степеней, расширенный граф степеней, граф независимости группы, ранговый граф группы, конечная группа, изоморфизм групп, π-разрешимая группа, простая группа, средний порядок элемента, разрешимая группа, дефицитный элемент, локально конечная группа, дистанционно регулярный граф, граф Крейна, сильно регулярный граф, граф Грюнберга–Кегеля (граф простых чисел), почти простая группа, граф Кэли, кликовый граф, реберно-транстранзитивный граф, нормальное накрытие графа, дважды транзитивный на дугах граф, полусимметричный граф, полный класс групп, ширина Бэра–Судзуки, симметрическая граница класса групп.

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The 2023 Ural Workshop on Group Theory and Combinatorics [1] was held online during the period 21 to 27 August 2023. The workshop was organized by the N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences [2], the Institute of Natural Sciences and Mathematics of the Ural Federal University [3], and the Ural Mathematical Center [4]. The 2023 Ural Workshop on Group Theory and Combinatorics continued the Ural Seminar on Group Theory and Combinatorics [5] and shared all declarations and rules of work of the seminar. The workshop covered modern aspects of group theory (including questions of actions of groups on combinatorial objects), graph theory, some combinatorial aspects of topology and optimization theory, and related topics, and aimed to support communications between specialists on group theory, combinatorics, and their applications from all over the world. The program of the workshop included 30 fifty-minute talks by keynote speakers and 21 twenty-minute contributed talks.

The first working day of the workshop (Monday, August 21) was mostly devoted to actions of groups on different combinatorial objects. Keynote speakers of the first day and their talks:

Cheryl Praeger (The University of Western Australia, Perth, Australia), “*Group theoretic constructions of normal covers of the complete bipartite graphs $K_{2^n,2^n}$*”.

Bernardo Rodrigues (University of Pretoria, Pretoria, South Africa), “*On finite imprimitive rank 3 permutation groups acting as permutation automorphism groups of self-dual code*”.

Gareth Jones (University of Southampton, Southampton, UK), "*Regular maps and hypermaps with primitive automorphism groups*".

Anton Klyachko (Lomonosov Moscow State University, Moscow, Russia), "*The cost of symmetry*".

Pablo Spiga (Department of Mathematics and Applications, University of Milano–Bicocca, Italy), "*Normal coverings of finite groups, the Isbell conjecture and the Erdős–Ko–Rado theorem*".

Also on Monday, August 21 there were 4 contributed talks.

The second working day of the workshop (Tuesday, August 22) was devoted to algorithms on groups, representations of groups, problems of coincidence of words in groups, and other questions. Keynote speakers of the second day and their talks:

Marston Conder (University of Auckland, Auckland, New Zeland), "*Finding subgroups of interest in finitely-presented groups*".

Sergey Gorchinskiy (Steklov Mathematical Institute of RAS, Moscow, Russia), "*Irreducible representations of finitely generated nilpotent groups*".

Mikhail Zaicev (Lomonosov Moscow State University, Moscow, Russia), "*On existence of PI-exponent of codimension growth*".

Cristina Acciarri (University of Modena and Reggio Emilia, Italy, and University of Brasilia, Brazil), "*On conciseness of words in residually finite groups*".

Gustavo Fernández-Alcober (University of the Basque Country, Bilbao, Spain), "*Concise words and conciseness on normal subgroups*".

Also on Tuesday, August 22 there were 3 contributed talks.

The third working day of the workshop (Wednesday, August 23) was mostly devoted to graphs defined on groups. Keynote speakers of the third day and their talks:

Peter Cameron (University of St Andrews, UK), "*What can graphs and algebraic structures say to each other?*"

Andrea Lucchini (University of Padova, Padova, Italy), "*Solubilizers in profinite groups*".

Carmine Monetta (University of Salerno, Fisciano (Salerno), Italy), "*Group nilpotency from a graph point of view*".

Lev Kazarin (P.G. Demidov Yaroslavl State University, Yaroslavl, Russia), "*Graphs, factorizations and the structure of a group*".

Alexandre Zalesskii (University of Brasilia, Brasilia, Brazil), "*Hall–Higman type theorems, Grunberg–Kegel graphs and fixed point elements in finite group representations*".

Also on Wednesday, August 23 there were 2 contributed talks.

The fourth working day of the workshop (Thursday, August 24) was mostly devoted to investigations on the subgroup structure and the normal structure of groups. Keynote speakers of the fourth day and their talks:

Tomasz Popiel (Monash University, Melbourne, Australia), "*The maximal subgroups of the Monster*".

Daria Lytkina (Sobolev Institute on Mathematics SB RAS, Novosibirsk State University, and Mathematical Center in Akademgorodok, Novosibirsk, Russia), "*Periodic Frobenius groups*".

Long Miao (Hohai University, Nanjing, China), "*On F-abnormal subgroups of finite groups*".

Marina Sorokina (I. G. Petrovsky Bryansk State University, Bryansk, Russia), "*Fibered and foliated formations of finite groups*".

Wenting Zhang (Lanzhou University, Lanzhou, China), "*Minimal non-finitely based involution monoids*".

Also on Thursday, August 24 there were 2 contributed talks.

The fifth working day of the workshop (Friday, August 25) was mostly devoted to arithmetical characterizations of finite groups. Keynote speakers of the fifth day and their talks:

Danila Revin (Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia and N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, Russia), "*On the Baer–Suzuki width of some radical classes*".

Maria Grechkoseeva (Sobolev Institute on Mathematics SB RAS and Novosibirsk State University, Novosibirsk, Russia), "*On characterization of finite groups by the set of element orders*".

Patrizia Longobardi (University of Salerno, Fisciano (Salerno), Italy), "*A new characterization of the alternating group of degree* 5".

Mercede Maj (University of Salerno, Fisciano (Salerno), Italy), "*On a problem related to the conjugacy classes of a group*".

Also on Friday, August 25 there were 3 contributed talks.

The sixth working day of the workshop (Saturday, August 26) was mostly devoted to algebraic combinatorics. Keynote speakers of the sixth day and their talks:

Yaokun Wu (Shanghai Jiao Tong University, Shanghai, China), "*Unavoidable intersection patterns of d-boxes*".

Alexander Makhnev (N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS and Ural Federal University, Yekaterinburg, Russia), "*On Krein graphs without triangles*".

Alexander Mednykh (Sobolev Institute on Mathematics SB RAS and Novosibirsk State University, Novosibirsk, Russia), "*Spectral invariants of cyclic covering of graphs and their applications in combinatorial analysis*".

Alexander Zvonkin (University of Bordeaux, Talence, France), "*Weighted trees*".

Also on Saturday, August 26 there were 6 contributed talks.

The last working day of the workshop (Sunday, August 27) was devoted to atrithmetical characterizations of finite groups and to Schur rings over groups. Keynote speakers of the last day of the workshop and their talks:

Marialaura Noce (University of Salerno, Fisciano (Salerno), Italy), "*On the product of element orders of finite group*".

Gang Chen (School of Mathematics and Statistics, Central China Normal University, Wuhan, China), "*Schur rings over free abelian group of rank two*".

Also on Sunday, August 27 there was 1 contributed talk. Moreover, the Open Problems Session was held at the end of the workshop.

Keeping with tradition, the participants of the workshop posed a number of open problems in their areas of research. Here we list the problems and some comments on them. The records of all discussions are available on the workshop website [1] for registered participants and on the website of the Ural Seminar on Group Theory and Combinatorics [5] after registration. The problems below are ordered so as to avoid repetition of definitions and notation.

1. Let G be a finite group. There are four graphs defined on the vertex set G. In each case the conditions for elements x and y to be adjacent are given below.

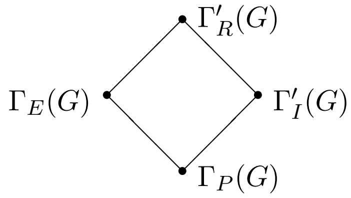
The *power graph* $\Gamma_P(G)$: $x = y^m$ or $y = x^m$ for some integer m.

The *enhanced power graph* $\Gamma_E(G)$: there exists z with $x = z^k$, $y = z^l$, for some integers k and l.

The *complement of the independence graph* $\Gamma'_I(G)$: there is no minimal (under inclusion) generating set containing $\{x, y\}$.

The *complement of the rank graph* $\Gamma'_R(G)$: there is no generating set of minimum cardinality containing $\{x, y\}$.

Under inclusion of edge sets, these are as follows.



The inclusions of Γ_P in Γ_E, and of Γ'_I in Γ'_R, are clear. For the other two, note that

if $x = y^m$, then we can delete x from any generating set containing $\{x, y\}$, so such a set cannot be minimal;

if $x = z^k$ and $y = z^l$, then we may replace x and y in any generating set containing $\{x, y\}$ by z, so such a set cannot have minimum cardinality.

- For every edge in the diagram, determine the groups G for which the two corresponding graphs are equal.

For Γ_P and Γ_E, these are the groups in which every element has prime power order; they were investigated by many authors, a summary of results can be found, for example, in [6, Theorem 1.7]. For Γ_P and Γ'_I, and for Γ_E and Γ'_R, they are determined in a recent paper by S. Freedman *et al.* [7]. The remaining case is open.

- In cases where the graphs are not equal, consider the difference graph on the group, whose edges are those of the larger graph which are not contained in the smaller. Investigate problems of these four difference graphs.

For the difference of Γ_E and Γ_P, work has begun, see [8]. In the other three cases, probably nothing has been done.

Peter Cameron

2. Does there exists a finite group G with two normal subgroups K and L, each having index 12 in G, such that

(a) $K \cong L$, and

(b) $G/K \cong C_{12}$ while $G/L \cong Alt_4$?

by Gabriel Verret, via Marston Conder; see Problem 20.21 in [9]

Proposition (Marston Conder and Natalia Maslova). *Let G be a finite group of the smallest possible order with two normal subgroups K and L, each with index* 12 *in G, such that K is isomorphic to L, but G/K is isomorphic to C_{12}, while G/L is isomorphic to Alt_4. Let $M = KL$ and $N = L \cap K$. Then the following statements hold*:

(1) *There is no $\psi \in \mathrm{Aut}(G)$ with $\psi(K) = \psi(L)$, in particular, K cannot be conjugate to L in G.*

(2) *M is a 2-group with $|G : M| = 3$, $|G : N| = 48$ and $G/N \cong C_4 \times Alt_4$.*

Proof. If there is $\psi \in \mathrm{Aut}(G)$ with $\psi(K) = \psi(L)$, then

$$G/L \cong \psi(G)/\psi(L) = G/K;$$

a contradiction. In particular, it is clear that K cannot be conjugate to L in G. Thus, statement (1) holds.

For statement (2), it is clear that M and N are normal subgroups of G with

$$M/K = KL/K \cong L/(K \cap L) = L/N \text{ and } M/L = KL/L \cong K/(K \cap L) = K/N.$$

Also $K \neq L$, and therefore K and L are proper normal subgroups of M, and N is a proper normal subgroup of each of K and L. Moreover, because $M/L \lhd G/L \cong Alt_4$ it follows that $M/L \cong Alt_4$ or $C_2 \times C_2$, and so M has index 1 or 3 in G.

Next, to show that $M \neq G$, suppose first that $KL = M = G$. Then

$$K/N \cong M/L = G/L \cong Alt_4 \text{ and } L/N \cong M/K = G/K \cong C_{12}$$

(and $G/N \cong C_{12} \times Alt_4$). Then because K is isomorphic to L, we find that L contains an isomorphic copy of N, say R, such that $L/R \cong Alt_4$, but then L contains isomorphic normal subgroups N and R with $L/N \cong C_{12}$ and $L/R \cong Alt_4$, and therefore L is a smaller counter-example than G, which is a contradiction. Thus $|G : M| = 3$, $M/K \cong C_4$ and $M/L \cong C_2 \times C_2$, and then also $K/N \cong M/L \cong C_2 \times C_2$ while $L/N \cong M/K \cong C_4$. In particular, it also follows that $|G/N| = |G/M||M/K||K/N| = 48$, and that $G/N \cong Alt_4 \times C_4$ and $M/N \cong C_2 \times C_2 \times C_4$.

Finally, to show that M is a 2-group, assume the contrary, and let p be any odd prime divisor of $|M|$, and let T be the non-trivial characteristic subgroup of M generated by all of its Sylow p-subgroups. Since $N \lhd M$ and M/N is an abelian 2-group, every Sylow p-subgroup of one of the subgroups K, L, M and N is a Sylow p-subgroup of each of the other three, and so T is generated by all of the Sylow p-subgroups of K, and by all of the Sylow p-subgroups of L. Hence, every isomorphism from K to L preserves T. But now G/T contains isomorphic normal subgroups K/T and L/T, with $(G/T)/(K/T) \cong G/K \cong C_{12}$ and $(G/T)/(L/T) \cong G/L \cong Alt_4$, again contradicting the minimality of G. Thus, statement (2) holds. $\square$

3. Let π be a set of primes and π' be the complement to π in the set $\mathbb{P}$ of all primes. If G is a finite group with a series of a normal subgroups

$$N_0 = \{1\} \leq N_1 \leq N_2 \leq \cdots \leq N_s = G,$$

where every factor group N_i/N_{i-1} is either a π'-group, or a p-group for some $p \in \pi$, then G is called a *π-solvable group.*

- Assume that π is a proper subset of $\mathbb{P}$. Let $G = AB = AC = BC$ be the product of proper π-solvable subgroups A, B and C of a finite simple non-abelian group G. What is G?

There are at least two examples of this situation.

(1) $G = Alt_6$ is an alternating group of degree 6. There are subgroups $A \cong Alt_5$, $B \cong Alt_5$ and C, which is the normalizer in G of its Sylow 3-subgroup, such that $G = AB = AC = BC$. Note that A and B are not conjugate in G. If $\pi = \mathbb{P} \setminus \{2, 3, 5\}$, then G is a simple π-solvable group.

(2) $G = P\Omega_8^+(q)$ have subgroups A, B, and C, all isomorphic to $P\Omega_7(q)$, such that $G = AB = AC = BC$.

- Assume that π is a proper subset of $\mathbb{P}$. Classify all finite simple non-abelian groups G that are products $G = AB$ of proper π-solvable subgroups A and B of G.

Note that if $\pi = \{2\}$, then the set of all π-solvable groups coincides with the set of all solvable groups.

Lev Kazarin

4. Let G be a finite group. If $x \in G$, write $|x|$ the order of x. If $x, y \in G$, denote by $[x, y] = x^{-1}y^{-1}xy$ the commutator of x and y, and write $K(G) = \{[x, y] \mid x, y \in G\}$, so $G' = \langle K(G)\rangle$. Define the functions

$$o(G) = \frac{1}{|G|}\sum_{x\in G}|x|, \quad o_K(G) = \frac{1}{|K(G)|}\sum_{[x,y]\in K(G)}|[x, y]|.$$

Answering the question posed by E. Khukhro, A. Moretó and M. Zarrin in [10], Marcel Herzog, Patrizia Longobardi, and Mercede Maj proved in [11] and in [12] that if $o(G) \le o(Alt_5)$, then either G is solvable or $G \cong Alt_5$. Moreover, $o(G) = o(Alt_5)$ if and only if $G \cong Alt_5$.

- Are there other finite simple groups S such that $o(G) = o(S)$ if and only if $G \cong S$?
- If $o_k(G) \le o_k(Alt_5)$, is it true that either G is solvable or $G \cong Alt_5$? Since in a finite simple group every element is a commutator, we can assume that G is a non-simple group.

In [11] it has been proven that if G is a finite group containing a non-trivial normal subgroup N, then $o(G/N) < o(G)$.

- If N is a non-trivial normal subgroup of a finite group G, what is the relationship between $o_K(G)$ and $o_K(G/N)$?

Patrizia Longobardi and Mercede Maj

5. Let G be a group. An element x of $G \setminus \{1\}$ will be called *deficient* if

$$\langle x\rangle < C_G(x)$$

and it will be called *non-deficient* if

$$\langle x\rangle = C_G(x).$$

Obviously if x is a deficient (non-deficient) element, then every element in the conjugacy class x^G of x in G is deficient (non-deficient). If $x \in G$ is deficient (non-deficient), then the conjugacy class x^G will be also called *deficient* (*non-deficient*).

Let j be a non-negative integer. We shall say that the group G *has defect* j, denoted by $G \in D(j)$ or by the phrase "*G is a $\mathcal{D}(j)$-group*", if exactly j non-trivial conjugacy classes of G are deficient.

In the paper [13], the authors studied $\mathcal{D}(j)$-groups G, $j \ge 1$, satisfying the following condition: G contains an element x of order p^{j+1}, for some prime p. This class of groups is denoted by $\mathcal{M}(j)$. In [13] it was proved that any group in $\mathcal{M}(j)$ is periodic, and the structure of all locally finite $\mathcal{M}(j)$-groups was determined.

Consider now the class of $D(j)$-groups G, where $j \ge 1$, satisfying the following condition: G contains an element x of order p^j, for some prime p. Call $\mathcal{N}(j)$ this class of groups.

- Is a group in $\mathcal{N}(j)$ periodic?

- What is the structure of a locally finite group in $\mathcal{N}(j)$?

Patrizia Longobardi and Mercede Maj

6. Let Γ be a simple graph and i be an integer with $1 \leq i \leq diam(\Gamma)$. Define Γ_i to be a simple graph with the same vertex set as Γ such that two different vertices are adjacent in Γ_i if and only if they are at distance i in Γ. It is clear that $\Gamma_1 = \Gamma$.

- Does there exist a distance-regular graph Γ such that Γ_2 and Γ_3 are strongly regular?
- Classify distance-regular graphs Γ of diameter 5 such that Γ_5 is also distance-regular of diameter 5.

Alexander Makhnev

7. A Krein graph $\Gamma = Kre(r)$ is a triangle-free strongly regular with parameters

$$((r^2+3r)^2, r^3+3r^2+r, 0, r^2+r).$$

For each vertex u, the induced subgraph $\Gamma_2(u)$ on the second neighborhood of the vertex u in Γ is also strongly regular with parameters

$$(r^4+5r^3+6r^2-r, r^3+2r^2, 0, r^2).$$

Denote this graph by $Kre(r)'$.

Let r be a positive integer. Does there exist

- A strongly regular graph $Kre(r)$?
- An extendable symmetric 2-$((s+2)(s^2+4s+2), (s^2+3s+1), s)$ design for $s = r-1$?
- A connected strongly regular graph such that its local subgraphs are complements to $Kre(r)'$?
- A distance-regular graph with intersection array

  $$\{r^3+3r^2+r, r^3+3r^2+r-1, r^3+2r^2, r^2+r, 1; 1, r^2+r, r^3+2r^2, r^3+3r^2+r-1, r^3+3r^2+r\}?$$

Alexander Makhnev

8. Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the *spectrum* of G, that is, the set of all element orders of G. The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of G; in this simple graph the vertex set is $\pi(G)$, and distinct vertices p and q are adjacent if and only if $pq \in \omega(G)$.

A finite group G is

recognizable by its Gruenberg–Kegel graph if for each finite group H, $\Gamma(G) = \Gamma(H)$ if and only if $G \cong H$;

k-recognizable by Gruenberg–Kegel graph, where k is a positive integer, if there are exactly k pairwise non-isomorphic finite groups H with $\Gamma(H) = \Gamma(G)$;

almost recognizable by Gruenberg–Kegel if it is k-recognizable Gruenberg–Kegel graph for some positive integer k;

unrecognizable by Gruenberg–Kegel graph, if there are infinitely many pairwise non-isomorphic finite groups H with $\Gamma(H) = \Gamma(G)$.

A finite group G is *almost simple* if $S \cong Inn(S) \trianglelefteq S \leq Aut(S)$ for a finite simple non-abelian group S.

In [6] me and Peter Cameron proved that if G is a finite group such that G is almost recognizable by Gruenberg–Kegel graph, then G is almost simple.

- Let G be an almost simple group. Decide whether G is recognizable, k-recognizable for some integer $k > 1$, or unrecognizable by its Gruenberg–Kegel graph.

At the moment there are a number of results on recognition of simple groups by Gruenberg–Kegel graph; some survey of known results obtained before 2022 can be found in [6]. Recently jointly with Viktor Panshin and Alexey Staroletov in [14] we proved that every simple exceptional group of Lie type, which is isomorphic to neither ${}^2B_2(2^{2n+1})$ with $n \geq 1$ nor $G_2(3)$ and whose Gruenberg–Kegel graph has at least three connected components, is almost recognizable by Gruenberg–Kegel graph; groups ${}^2B_2(2^{2n+1})$, where $n \geq 1$, and $G_2(3)$ are unrecognizable by Gruenberg–Kegel graph.

Natalia Maslova

9. Let p be an odd prime, H be a finite group with subgroups $X \cong Y \cong C_p^n$ such that $H = \langle X, Y \rangle$ and $H/H' \cong X \times Y$.

Let $C(H, X, Y)$ be the Cayley graph $C(H, S)$ with $S = (X \cup Y) \setminus \{1\}$.

Two families of maximal cliques (complete subgraphs) in $C(H, X, Y)$ are

$$\{Xh \mid h \in H\} \text{ and } \{Yh \mid h \in H\}.$$

These form the vertex set of its clique graph $\Sigma(H, X, Y)$ with edge set

$$E\Sigma = \{\{Xh, Yg\} \mid Xh \cap Yg \neq \varnothing\}.$$

- Is $\Sigma(H, X, Y)$ always an edge-transitive H-normal cover of $\Sigma_{H'} = K_{p^n,p^n}$?
- Under what conditions is $\Sigma(H, X, Y)$ 2-arc-transitive?
- Under what conditions is $\Sigma(H, X, Y)$ semisymmetric?
- Under what conditions is $\Sigma(H, X, Y)$ a Cayley graph?
- Find explicit infinite families with some of these properties.

Solutions of corresponding problems for $p = 2$ can be found in [15].

Cheryl Praeger, Daniel Hawtin and Jin-Xin Zhou

10. Let $\mathfrak{X}$ be a non-empty class of groups closed under taking

homomorphic images (if $G \in \mathfrak{X}$ and ϕ is a homomorphism with domain G, then $G^\phi \in \mathfrak{X}$),

subgroups (if $G \in \mathfrak{X}$ and $H \leqslant G$, then $H \in \mathfrak{X}$), and

(the Fitting property) products of normal subgroups ($H, K \trianglelefteq G$ and $H, K \in \mathfrak{X}$, then $HK \in \mathfrak{X}$).

Then, in every group G, there is the $\mathfrak{X}$-*radical*, i. e. the largest normal $\mathfrak{X}$-subgroup

$$G_{\mathfrak{X}} := \langle H \mid H \trianglelefteq G \text{ and } H \in \mathfrak{X} \rangle.$$

According to [16], the *Baer–Suzuki width of* $\mathfrak{X}$ is $\mathrm{BS}(\mathfrak{X}) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ such that $\mathrm{BS}(\mathfrak{X}) \leqslant m$ for $m \in \mathbb{N} \cup \{0\}$ if and only if

$$G_{\mathfrak{X}} = \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathfrak{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}$$

for every group G. If such an $m \in \mathbb{N} \cup \{0\}$ does not exist then $\mathrm{BS}(\mathfrak{X}) = \infty$.

- Let $\mathfrak{X}$ satisfies the above assumptions. Is the Baer–Suzuki width of $\mathfrak{X}$ finite?

According to [17], a non-empty class $\mathfrak{X}$ of groups is said to be *complete* if $\mathfrak{X}$ is closed under taking

homomorphic images (if $G \in \mathfrak{X}$ and ϕ is a homomorphism with domain G, then $G^{\phi} \in \mathfrak{X}$),

subgroups ($G \in \mathfrak{X}$ and $H \leqslant G$, then $H \in \mathfrak{X}$), and

extensions ($H \trianglelefteq G$ and $H, G/H \in \mathfrak{X}$, then $G \in \mathfrak{X}$).

Danila Revin has announced in his talk that the Baer–Suzuki width of a complete class $\mathfrak{X}$ is always finite.

Danila Revin

11. The *symmetric boundary* of a non-empty class $\mathfrak{X}$ of groups (denoted by $\Upsilon(\mathfrak{X})$) is the largest integer n such that $Sym_n \in \mathfrak{X}$. Let $\Upsilon(\mathfrak{X}) := \infty$ if $Sym_n \in \mathfrak{X}$ for all positive integers n.

Danila Revin has announced in his talk that for a complete class $\mathfrak{X}$ of groups, if $\Upsilon(\mathfrak{X}) < \infty$, then

$$\Upsilon(\mathfrak{X}) \leqslant \mathrm{BS}(\mathfrak{X}) \leqslant \max\{11, 2\Upsilon(\mathfrak{X}) + 1\}.$$

- Find the best possible function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{BS}(\mathfrak{X}) \leq f(\Upsilon(\mathfrak{X}))$ for each complete class $\mathfrak{X}$ of groups with $\Upsilon(\mathfrak{X}) < \infty$.

Danila Revin

The work of the 2023 Ural Workshop on Group Theory and Combinatorics will be continued by the work of the Ural Seminar on Group Theory and Combinatorics, which will be held every other Tuesday with possible changes and exceptions. The list of talks of the seminar can be found on its website [5].

We are looking forward to meet you at the Ural Seminar on Group Theory and Combinatorics!

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