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POWER DEGREES IN DYNAMIC MULTI-AGENT SYSTEMS[1]

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Dynamic multi-agent systems connected in network are considered. To define the power of each agent the analogue of characteristic function is introduced. The values of this characteristic function for each coalition (subset of agents) are calculated as joint payoff of players from this coalition plus payoffs (multiplied on some discount factor) of players which do not belong to the coalition S but have connections with players from S. We suppose that the dynamic of the system is prescribed (this maybe cooperation, Nash equilibrium or any other behaviour). Thus, the characteristic function is evaluated along the prescribed trajectory of agents. And it measures the worth of coalitions under the motion along this trajectory instead of under minimax confrontation or the Nash non-cooperative stance. As solution we consider the proportional solution and introduce Power degrees of an agent based on proportional solution. It is shown that the Power degree (PD) belongs to the Core. PD rank agents according to their importance.

Keywords: multi-agent system and proportional solution and power degree.

Л. А. Петросян, Д. В. К. Янг, Я. Б. Панкратова. Индекс значимости в динамических многоагентных системах.

Рассматриваются динамические мультиагентные системы на сети. Для определения силы игрока вводится аналог характеристической функции. Значения этой характеристической функции для каждой коалиции (подмножества агентов) рассчитываются как совместный выигрыш игроков (агентов) из этой коалиции при движении вдоль предписанной заранее траектории плюс выигрыши, умноженные на некоторый коэффициент дисконтирования, игроков (агентов), которые не принадлежат коалиции S, но имеют связи с игроками из S. Предполагается, что динамика системы предписана заранее (это может быть кооперативной поведение, движение в равновесии по Нэшу, или какой либо другое движение). Характеристическая функция, вычисляемая вдоль предписанной траектории агентов, измеряет значимость коалиций при движении вдоль этой траектории, а не в условиях минимаксного подхода или равновесия по Нэшу. В качестве решения мы рассматриваем пропорциональное решение и вводим понятие индекса значимости агента, основанное на пропорциональном решении. Вектор, составленный из индексов значимости, ранжирует агентов в соответствии с их важностью. Показано, что вектор, составленный из индексов значимости агентов, принадлежит C-ядру. Исследуется вопрос устойчивости ранжирования агентов при развитии мультиагентной системы вдоль предписанной траектории.

Ключевые слова: мультиагентная система, пропорциональное решение и индекс значимости.

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Recently, many interesting problems have been modelled using differential games on networks. The first pioneering papers in this field are [1–5]. An obvious continuation of research in the field of dynamic games is to extend them to the class of cooperative dynamic games on networks (the following papers should be noted [6], and the paper of [7–9]). Different properties of the cooperative solutions of dynamic network games are investigated in [10]. In the paper [11], the differential games on networks with partner sets are considered. In this paper, we consider cooperative behaviour only as one of possible behaviours of agents. We suppose that multi-agent system develops along the prescribed trajectory (in some cases, this trajectory can coincide with cooperative or Pareto optimal, or NE trajectory). The values of characteristic function for each subset of agents (coalition) are calculated as joint payoff of agents from this coalition plus payoffs (multiplied on discount factor depending from S) of agents which do not belong to the coalition S but have connections with agents from S. This setting can be interpreted as a realisation of sanctions by agents from $N\backslash S$

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against agents from S. The characteristic function is evaluated along the prescribed trajectory. And it measures the worth of coalitions under motion along this trajectory instead of under minimax confrontation or the Nash non-cooperative stance. It can be seen that the new characteristic function is superadditive under some conditions. In this paper, we introduce the Power degree of an agent based on proportional solution.

1. Dynamic Multi-Agent System Connected in Network

Consider a class of n-person dynamic multi-agent systems on network with time horizon $[t_0, T]$. The agents are connected in a network. We use $N = \{1, 2, \cdots, n\}$ to denote the set of agents in the network. The nodes of the network are used to represent the agents from the set N. We also denote the set of nodes by N and denote the set of all arcs in network N by L. The arcs in L are the $arc\ (i, j) \in L$ for agents $i, j \in N$, $i \neq j$. For notational convenience, we denote the set of agents connected to agent i as $\tilde{K}(i) = \{j : arc(i, j) \in L\}$, for $i \in N$.

Let $x^i(\tau) \in \mathbb{R}^m$ be the state variable of agent $i \in N$ at time τ, and $u^i(\tau) \in U^i \subset \mathbb{R}^k$ the control variable of agent representing his communication efforts $i \in N$.

In previous models [12; 8], it was supposed that agent $i \in N$ can cut connection with any other agent from the set $\tilde{K}(i)$ at any instant of time. In this paper, we exclude this possibility.

The state dynamics of the system is

$$\dot{x}^i(\tau) = f^i(x^i(\tau), u^i(\tau)), \quad x^i(t_0) = x^i_0, \tag{1}$$

for $\tau \in [t_0, T]$ and $i \in N, u^i(\tau) \in U^i \subset Comp\,\mathbb{R}^l$.
The function $f^i(x^i, u^i)$ is continuously differentiable in x^i and u^i.

The payoff function of agent i depends upon his state variable and the state variables of agents from the sets $\tilde{K}(i)$.

In particular, the payoff of agent i is given as

$$H_i(x^1_0, \ldots, x^n_0, u^1, \ldots, u^n) = \sum_{j \in \tilde{K}(i)} \int_{t_0}^{T} h^j_i(x^i(\tau), x^j(\tau))d\tau. \tag{2}$$

The term $h^j_i(x^i(\tau), x^j(\tau))$ is the instantaneous gain that agent i can obtain through communication with agent $j \in \tilde{K}(i)$ (note that the pair $(i, i) \notin L$). The functions $h^j_i(x^i(\tau), x^j(\tau))$, for $j \in \tilde{K}(i)$ are non-negative and continuous. For notational convenience, we use $x(t)$ to denote the vector $(x^1(t), x^2(t), \cdots, x^n(t))$.

2. Characteristic Function

In this section, we introduce a new type of characteristic function which differs from one defined in paper [6] and after used in [12; 8].

Suppose that agents for some reason decide to move along prescribed trajectory subject to dynamics (1).

Denote this trajectory by $\bar{x}(t) = (\bar{x}^1(t), \bar{x}^2(t), \cdots, \bar{x}^n(t))$ and corresponding controls $\bar{u}(t) = (\bar{u}^1(t), \bar{u}^2(t), \cdots, \bar{u}^n(t))$.

The joint payoff involving all agents will be expressed as

$$\sum_{i \in N} \left(\sum_{j \in \tilde{K}(i)} \int_{t_0}^{T} h^j_i(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau \right) = V(N; x_0, T - t_0) \tag{3}$$

subject to dynamics (1).

For each $S \subset N$ define the characteristic function which represent the worth the coalition $S \subset N$ as

$$V(S;x_0,T-t_0) = \sum_{i\in S}\Bigg[\sum_{j\in\tilde{K}(i)\cap S}\int_{t_0}^{T} h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau))d\tau \;+\; \alpha(S)\sum_{j\in\tilde{K}(i)\cap N\setminus S}\int_{t_0}^{T} h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau))d\tau\Bigg], \quad (4)$$

$\alpha(S) \in (0,1)$ is the discount factor, and the quantity $(1-\alpha(S))$ can be interpreted as sanction level used by coalition $N\setminus S$ against coalition S.

Note that the worth of coalition S is measured by the sum of payoffs of agents in the coalition when the process develops along the prescribed trajectory $\bar{x}(t) = (\bar{x}^1(t), \bar{x}^2(t), \cdots, \bar{x}^n(t))$ plus the sum of payoffs (multiplied on discount factor $\alpha(S)$) of agents which do not belong to the coalition S but have connections with agents from S. This shows that players from opposite coalition $N\setminus S$ acting against coalition S decrease the income of coalition S.

For simplicity in notation, we denote

$$\beta_{ij}(x_0, T-t_0) = \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau))d\tau, \quad (5)$$

$$\beta_{ij}(\bar{x}(t), T-t) = \int_{t}^{T} h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau))d\tau, \quad t\in[t_0,T]. \quad (6)$$

Using notations (5), (6), we can express (4) as

$$V(S;x_0,T-t_0) = \sum_{i\in S}\Bigg[\sum_{j\in\tilde{K}(i)\cap S}\beta_{ij}(x_0,T-t_0) + \alpha(S)\sum_{j\in\tilde{K}(i)\cap N\setminus S}\beta_{ij}(x_0,T-t_0)\Bigg]. \quad (7)$$

Suppose that the following condition holds $\alpha(S_1) \le \alpha(S_2)$, for $S_1 \subset S_2$, $S_1 \subset N$, $S_2 \subset N$. This condition means that the sanction level $(1-\alpha(S))$ is decreasing when the size of coalition S increases. It can be understood from practical view point since sanction against large coalitions are usually less effective than against small ones.

Proposition 1. *Under above conditions, the characteristic function defined by* (4) *is superadditive.*

Proof. Prove the following inequalities:

$$V(S_1;x_0,T-t_0) + V(S_2;x_0,T-t_0) \le V(S_1\cup S_2;x_0,T-t_0) \quad (8)$$

where $S_1 \cap S_2 = \varnothing$.

Using (4), we have

$$V(S_1;x_0,T-t_0) \;+\; V(S_2;x_0,T-t_0)$$
$$= \sum_{i\in S_1}\Bigg[\sum_{j\in\tilde{K}(i)\cap S_1}\beta_{ij}(x_0,T-t_0) \;+\; \alpha(S_1)\sum_{j\in\tilde{K}(i)\cap N\setminus S_1}\beta_{ij}(x_0,T-t_0)\Bigg]$$
$$+\sum_{i\in S_2}\Bigg[\sum_{j\in\tilde{K}(i)\cap S_2}\beta_{ij}(x_0,T-t_0) \;+\; \alpha(S_2)\sum_{j\in\tilde{K}(i)\cap N\setminus S_2}\beta_{ij}(x_0,T-t_0)\Bigg].$$

In the square brackets of previous formula we have the payoff of player i, when $i \in S_1$ or $i \in S_2$. Denote it by $K_i(S)$. Consider the payoff of player i, $i \in S_1$

$$K_i(S_1) = \sum_{j\in\tilde{K}(i)\cap S_1}\beta_{ij}(x_0,T-t_0) \;+\; \alpha(S_1)\sum_{j\in\tilde{K}(i)\cap N\setminus S_1}\beta_{ij}(x_0,T-t_0)$$

and compare this payoff with payoff of player i if $i \in S_1 \cup S_2$

$$K_i(S_1 \cup S_2) = \sum_{j \in \tilde{K}(i) \cap (S_1 \cup S_2)} \beta_{ij}(x_0, T-t_0) \;+\; \alpha(S_1 \cup S_2) \sum_{j \in \tilde{K}(i) \cap N \setminus (S_1 \cup S_2)} \beta_{ij}(x_0, T-t_0)$$

$$= \sum_{j \in \tilde{K}(i) \cap S_1} \beta_{ij}(x_0, T-t_0) + \sum_{j \in \tilde{K}(i) \cap S_2} \beta_{ij}(x_0, T-t_0) + \alpha(S_1 \cup S_2) \sum_{j \in \tilde{K}(i) \cap N \setminus (S_1 \cup S_2)} \beta_{ij}(x_0, T-t_0).$$

Since $S_1 \cap S_2 = \varnothing$ we can rewrite $K_i(S_1)$:

$$K_i(S_1) = \sum_{j \in \tilde{K}(i) \cap S_1} \beta_{ij}(x_0, T-t_0)$$

$$+ \alpha(S_1) \sum_{j \in \tilde{K}(i) \cap S_2} \beta_{ij}(x_0, T-t_0) + \alpha(S_1) \sum_{j \in \tilde{K}(i) \cap N \setminus (S_1 \cup S_2)} \beta_{ij}(x_0, T-t_0) \le \sum_{j \in \tilde{K}(i) \cap S_1} \beta_{ij}(x_0, T-t_0)$$

$$+ \; \alpha(S_1 \cup S_2) \sum_{j \in \tilde{K}(i) \cap S_2} \beta_{ij}(x_0, T-t_0) \;+\; \alpha(S_1 \cup S_2) \sum_{j \in \tilde{K}(i) \cap N \setminus (S_1 \cup S_2)} \beta_{ij}(x_0, T-t_0) \tag{9}$$

$$\le \sum_{j \in \tilde{K}(i) \cap S_1} \beta_{ij}(x_0, T-t_0)$$

$$+ \sum_{j \in \tilde{K}(i) \cap S_2} \beta_{ij}(x_0, T-t_0) \;+\; \alpha(S_1 \cup S_2) \sum_{j \in \tilde{K}(i) \cap N \setminus (S_1 \cup S_2)} \beta_{ij}(x_0, T-t_0) = K_i(S_1 \cup S_2),$$

for any $i \in S_1$, similarly (9) holds for $i \in S_2$. And we get that each summand in the left side of (8) is less or equal to the corresponding summand in the right side of (8).

Hence proposition is proved. □

3. Special Type of Imputations

In this section, we introduce the proportional solution using the defined characteristic function.

Now, we consider allocating the grand coalition cooperative network gain $V(N; x_0, T-t_0)$ along prescribed trajectory to individual players according to the the proportional solution.

To define the proportional solution compute first the values $V(\{i\})$ for $i = 1, \ldots, n$ (the case when $S = \{i\}$).

$$V(\{i\}; x_0, T-t_0) = \alpha(\{i\}) \cdot \sum_{j \in \tilde{K}(i)} \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau, \tag{10}$$

since $\tilde{K}(i) \cap N = \tilde{K}(i)$. The proportional solution $\{p_i(x_0, T-t_0)\}$ can be written in the form

$$p_i(x_0, T-t_0) = \frac{V(\{i\}; x_0, T-t_0)}{\sum\limits_{i \in N} V(\{i\}; x_0, T-t_0)} \cdot V(N; x_0, T-t_0)$$

$$= \frac{\alpha(\{i\}) \sum\limits_{j \in \tilde{K}(i)} \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau}{\sum\limits_{i \in N} \alpha(\{i\}) \sum\limits_{j \in \tilde{K}(i)} \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau} \cdot V(N; x_0, T-t_0). \tag{11}$$

If we consider the special case when $\alpha(\{i\}) = \alpha$, $i \in N$, using (3) we get

$$p_i(x_0, T-t_0) = \sum_{j \in \tilde{K}(i)} \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau.$$

Proposition 2. *Proportional solution is time-consistent and belongs to the Core of the corresponding cooperative game (the case, when $\alpha(\{i\}) = \alpha$, $i = 1, \ldots, n$).*

Proof.

$$p_i(x_0, T - t_0) = \sum_{j \in \tilde{K}(i)} \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau$$

$$= \sum_{j \in \tilde{K}(i)} \int_{t_0}^{t} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau + \sum_{j \in \tilde{K}(i)} \int_{t}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau \tag{12}$$

$$= \sum_{j \in \tilde{K}(i)} \int_{t_0}^{t} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau + p_i(\bar{x}(t), T - t).$$

This shows time-consistency of the proportional solution.

Also for any coalition $S \subset N$ we have

$$\sum_{i \in S} p_i(x_0, T - t_0) = \sum_{i \in S} \sum_{j \in \tilde{K}(i)} \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau$$

$$\geq \sum_{i \in S} \sum_{j \in \tilde{K}(i) \cap S} \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau = V(S; x_0, T - t_0). \tag{13}$$

Inequality (13) shows that vector $(p_1(x_0, T - t_0), \ldots, p_n(x_0, T - t_0))$ belongs to the Core [8].

Hence Proposition 2 follows. □

4. Power Degree of an Agent Based on Proportional Solution

Network interactions in time were considered in [3;13]. Unlike the [6;7] here we will not assume that agents seek to somehow optimize their influence, but we assume that the system develops in a certain way and we will only investigate the issue of its stability from the point of view of preserving the influence of agents in the process. To determine the degree of influence of agents, one can use both the Shapley value or the proportional solution. In this paper, we shall use the proportional solution.

As we defined before the multi-agent system develops along a prescribed trajectory $\bar{x}(t)$, $t \in [t_0, T]$. In each time instant $t \in [t_0, T]$ agents find themselves in a point $\bar{x}(t)$ on this trajectory. Thus, the Power degree of an agent based on proportional solution at time $t \in [t_0, T]$ will be equal (the case $\alpha(\{i\}) = \alpha,\ i = 1, \ldots, n$)

$$p_i(\bar{x}(t), T - t) = \sum_{j \in \tilde{K}(i)} \int_{t}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau.$$

Suppose that agents are numerated such way that

$$p_1(x_0, T - t_0) \geq p_2(x_0, T - t_0) \geq \ldots \geq p_n(x_0, T - t_0). \tag{14}$$

Consider now Power degree of agents at time instant $t \in [t_0, T]$

$$p_i(\bar{x}(t), T - t), \quad i \in N,$$

when the system develops along prescribed trajectory $\bar{x}(t)$ under control $\bar{u}(t)$. We shall call the trajectory (development) $\bar{x}(t)$ stable if

$$p_1(\bar{x}(t), T-t) \geq p_2(\bar{x}(t), T-t) \geq \ldots \geq p_n(\bar{x}(t), T-t), \quad t \in [t_0, T]. \tag{15}$$

The stable development (trajectory) is a rare event. It is sometimes important that (15) holds for a some kind of optimal trajectories (cooperative trajectory, Pareto optimal trajectory, NE trajectory and etc.)

It is clear that if the functions $h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) = \bar{\bar{h}}_i^j = const$ (do not depend on time) the condition (15) will always hold. Since in this case

$$p_i(\bar{x}(t), T-t) = \sum_{j \in \tilde{K}(i)} \bar{\bar{h}}_i^j (T-t) = (T-t) \sum_{j \in \tilde{K}(i)} \bar{\bar{h}}_i^j.$$

And the condition (15) will be equivalent to

$$\sum_{j \in \tilde{K}(1)} \bar{\bar{h}}_1^j \geq \sum_{j \in \tilde{K}(2)} \bar{\bar{h}}_2^j \geq \ldots \geq \sum_{j \in \tilde{K}(n)} \bar{\bar{h}}_n^j.$$

If $\bar{\bar{h}}_i^j = 1$, then $p_i(\bar{x}(t), T-t)$ is proportional to the number of arcs which connect agent i with other agents from network N. Then the introduced Power degree coincides to one considered earlier [14], and it is of course stable.

One can see that stability condition (15) can be satisfied if we could change instantaneous payments $h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))$, $\tau \in [t_0, T]$, $i \in N$, $j \in \tilde{K}(i)$ preserving the total payment

$$\int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau.$$

For this reason, we can use the classical mean value theorem

$$\int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau = \int_{t_0}^{T} h_i^j(\theta_{ij}) d\tau = h_i^j(\theta_{ij})(T - t_0), \;\; \theta_{ij} \in [t_0, T].$$

Then the condition (14) can be written as

$$\sum_{j \in \tilde{K}(1)} h_1^j(\theta_{1j}) \geq \sum_{j \in \tilde{K}(2)} h_2^j(\theta_{2j}) \geq \ldots \geq \sum_{j \in \tilde{K}(n)} h_n^j(\theta_{nj}).$$

It is clear that if we replace stage payments $h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau))$ by $h_i^j(\theta_{ij})$ on time interval $[t_0, T]$ the stability condition (15) will hold on the whole time interval of multi-agent system development.

The same approach can be used if as Power degree we shall consider the Shapley value.

5. Examples

Example 1. Consider the following 3 player network game (see Fig. 1).

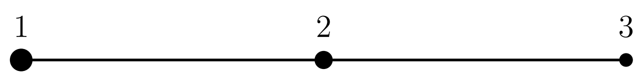

Fig. 1. Three player network game.

Denote for convenience $V(S;x_0,T-t_0)$ as $V(S)$, $S\subset N$, $\beta_{ij}(x_0.T-t_0)$ as β_{ij}, and $\beta_{ij}(\bar{x}(t),T-t)=\beta_{ij}(t)$. For this network structure the values of characteristic function are defined as

$$V(\{1\})=\alpha(\{1\})\beta_{12},\quad V(\{2\})=\alpha(\{2\})(\beta_{21}+\beta_{23}),\quad V(\{3\})=\alpha(\{3\})\beta_{32},$$

$$V(\{1,2\})=\beta_{12}+\beta_{21}+\alpha(\{1,2\})\beta_{23},\quad V(\{1,3\})=\alpha(\{1,3\})(\beta_{32}+\beta_{12}),$$

$$V(\{2,3\})=\beta_{23}+\beta_{32}+\alpha(\{2,3\})\beta_{21},\quad V(\{1,2,3\})=\beta_{12}+\beta_{21}+\beta_{23}+\beta_{32}.$$

Computing the proportional solution using (10) we have

$$p_1(x_0,T-t_0)=\frac{\alpha(\{1\})\beta_{12}}{\alpha(\{1\})\beta_{12}+\alpha(\{2\})(\beta_{21}+\beta_{23})+\alpha(\{3\})\beta_{32}}(\beta_{12}+\beta_{21}+\beta_{23}+\beta_{32}),$$

$$p_2(x_0,T-t_0)=\frac{\alpha(\{2\})(\beta_{21}+\beta_{23})}{\alpha(\{1\})\beta_{12}+\alpha(\{2\})(\beta_{21}+\beta_{23})+\alpha(\{3\})\beta_{32}}(\beta_{12}+\beta_{21}+\beta_{23}+\beta_{32}),$$

$$p_3(x_0,T-t_0)=\frac{\alpha(\{3\})\beta_{32}}{\alpha(\{1\})\beta_{12}+\alpha(\{2\})(\beta_{21}+\beta_{23})+\alpha(\{3\})\beta_{32}}(\beta_{12}+\beta_{21}+\beta_{23}+\beta_{32}).$$

To make a conclusion about stability of prescribed trajectory we should compare the power degrees of agents at each time instant t, $t\in[t_0,T]$ according to proportional solution.

To understand the relation between components of proportional solution it is sufficient to consider the relations between the numerator

$$\alpha(\{1\})\beta_{12},\ \alpha(\{2\})(\beta_{21}+\beta_{23}),\ \alpha(\{3\})\beta_{32}.$$

In this example, we shall suppose that instantaneous payoffs $h_i^j(\bar{x}^i(\tau),\bar{x}^j(\tau))$ are constant (do not depend on time), and are equal to

$$h_1^2(t)=2,\ h_2^1(t)=3, h_2^3(t)=4,\ h_3^2(t)=5,\ t\in[t_0,T],$$

then

$$p_1(\bar{x}(t),T-t)=\frac{2\alpha(\{1\})}{2\alpha(\{1\})+7\alpha(\{2\})+5\alpha(\{3\})}\cdot 14(T-t),$$

$$p_2(\bar{x}(t),T-t)=\frac{7\alpha(\{2\})}{2\alpha(\{1\})+7\alpha(\{2\})+5\alpha(\{3\})}\cdot 14(T-t),$$

$$p_3(\bar{x}(t),T-t)=\frac{5\alpha(\{3\})}{2\alpha(\{1\})+7\alpha(\{2\})+5\alpha(\{3\})}\cdot 14(T-t).$$

In case $\alpha(\{1\})\geq\frac{7}{2}\alpha(\{2\})$, $\alpha(\{1\})\geq\frac{5}{2}\alpha(\{3\})$ and $\alpha(\{2\})\geq\frac{5}{7}\alpha(\{3\})$ we will have the following relation between Power degrees $p_1(\bar{x}(t),T-t)\geq p_2(\bar{x}(t),T-t)\geq p_3(\bar{x}(t),T-t)$ for any $t\in[t_0,T]$, which means the stability of Power degree.

In special case, when $\alpha(\{i\})=\alpha$, $i=1,\ldots,n)$, we get

$$p_1(\bar{x}(t),T-t)=2(T-t),\quad p_2(\bar{x}(t),T-t)=7(T-t),\quad p_3(\bar{x}(t),T-t)=5(T-t).$$

Calculate the Power degrees using the Shapley value [15]

$$Sh_1(x_0,T-t_0)=\frac{\beta_{12}+\beta_{21}}{2}+\frac{\beta_{12}}{6}(2\alpha(\{1\})+\alpha(\{1,3\}))-\frac{\beta_{21}}{6}(2\alpha(\{2,3\})+\alpha(\{2\}))$$
$$+\frac{\beta_{23}}{6}(\alpha(\{1,2\})-\alpha(\{2\}))+\frac{\beta_{32}}{6}(\alpha(\{1,3\})-\alpha(\{3\})),$$

$$Sh_2(x_0, T-t_0) = \frac{\beta_{21}+\beta_{12}}{2} + \frac{\beta_{23}+\beta_{32}}{2} + \frac{\beta_{21}}{6}(2\alpha(\{2\}) + \alpha(\{2,3\}))$$
$$- \frac{\beta_{12}}{6}(2\alpha(\{1,3\}) + \alpha(\{1\})) - \frac{\beta_{32}}{6}(2\alpha(\{1,3\}) + \alpha(\{3\})) + \frac{\beta_{23}}{6}(\alpha(\{1,2\}) + 2\alpha(\{2\})),$$
$$Sh_3(x_0, T-t_0) = \frac{\beta_{23}+\beta_{32}}{2} + \frac{\beta_{32}}{6}(2\alpha(\{3\}) + \alpha(\{1,3\})) - \frac{\beta_{23}}{6}(2\alpha(\{1,2\}) + \alpha(\{2\}))$$
$$+ \frac{\beta_{12}}{6}(\alpha(\{1,3\}) - \alpha(\{1\})) + \frac{\beta_{21}}{6}(\alpha(\{2,3\}) - \alpha(\{2\})).$$

Using the values of instantaneous gains introduced above for prescribed trajectory $\bar{x}(t)$, $t \in [t_0, T]$, and suppose that $\alpha(S) = \alpha$, $S \subset N$ find the Shapley value in time instant $t \in [t_0, T]$

$$Sh_1(\bar{x}(t), T-t) = \frac{5-\alpha}{2}(T-t), \quad Sh_2(\bar{x}(t), T-t) = 7(T-t), \quad Sh_3(\bar{x}(t), T-t) = \frac{9+\alpha}{2}(T-t).$$

It is interesting that for the second player Power degree which corresponds to the proportional solution coincides with Power degree which corresponds to the Shapley value for any α.

Example 2. Consider the following 5 player network game (see Fig. 2).

In this case, we shall consider only proportional solution, and as result the following values of characteristic function

$$V(\{1\}) = \alpha(\{1\})(\beta_{12} + \beta_{14} + \beta_{15}), \quad V(\{2\}) = \alpha(\{2\})(\beta_{21} + \beta_{24} + \beta_{23}),$$
$$V(\{3\}) = \alpha(\{3\})\beta_{32}, \quad V(\{4\}) = \alpha(\{4\})(\beta_{41} + \beta_{42} + \beta_{45}), \quad V(\{5\}) = \alpha(\{5\})(\beta_{51} + \beta_{54}),$$
$$V(\{1,2,3,4,5\}) = \beta_{12} + \beta_{21} + \beta_{14} + \beta_{41} + \beta_{15} + \beta_{51} + \beta_{23} + \beta_{32} + \beta_{24} + \beta_{42} + \beta_{45} + \beta_{54}.$$

Suppose that the instantaneous gains $h_i^j(\bar{x}_i(t), \bar{x}_j(t))$ are constant on prescribed trajectory $\bar{x}(t)$, i.e. $h_i^j(\bar{x}_i(t), \bar{x}_j(t)) = h_i^j$ and take the following values

$$h_1^2 = 2, \quad h_2^1 = 3, \quad h_1^5 = 2, \quad h_5^1 = 3, \quad h_1^4 = 3, \quad h_4^1 = 4,$$
$$h_2^3 = 5, \quad h_3^2 = 6, \quad h_2^4 = 5, \quad h_4^2 = 3, \quad h_4^5 = 2, \quad h_5^4 = 2.$$

Compute the proportional solution using (10) in special case when $\alpha(\{i\}) = \alpha$, $i = 1, \ldots, 5$.

$$p_1(x_0, T-t_0) = \beta_{12} + \beta_{14} + \beta_{15} = (2+3+2)(T-t_0) = 7(T-t_0),$$
$$p_2(x_0, T-t_0) = \beta_{21} + \beta_{24} + \beta_{23} = (3+5+5)(T-t_0) = 13(T-t_0),$$
$$p_3(x_0, T-t_0) = \beta_{32} = 6(T-t_0),$$
$$p_4(x_0, T-t_0) = \beta_{41} + \beta_{42} + \beta_{45} = (4+3+2)(T-t_0) = 9(T-t_0),$$
$$p_5(x_0, T-t_0) = \beta_{51} + \beta_{54} = (3+2)(T-t_0) = 5(T-t_0).$$

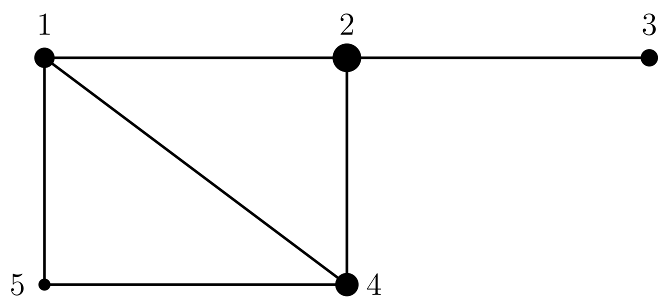

Fig. 2. Five player network game.

It is obvious that for prescribed trajectory $\bar{x}(t)$ the system of inequalities

$$p_5(\bar{x}(t), T-t) < p_3(\bar{x}(t), T-t) < p_1(\bar{x}(t), T-t) < p_4(\bar{x}(t), T-t) < p_2(\bar{x}(t), T-t)$$

is fulfilled for any time instant $t \in [t_0, T]$. This gives us the stability of the Power degree.

Conclusion

A novel form for measuring the worth of coalitions of agents in multi-agent dynamic system is developed. In computing this type of characteristic function, the values for each coalition are calculated as joint payoff of players from this coalition plus payoffs (multiplied on some discount factor) of players which do not belong to the coalition S but have connections with players from S along a given (prescribed) trajectory. Thus, the values of characteristic function for coalition S take into account the influence of players which are not in coalition S, but this does not make calculations harder as in our previous papers [9;11;12]. The condition is derived for the supperadditivity of defined characteristic function. The concept of Power degrees of an agent is introduced. The stability of the PD is defined and shown in special case.

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