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Uniform with Respect to the Parameter $a \in (0, 1)$ Two-Sided Estimates of the Sums of Sine and Cosine Series with Coefficients $1/k^a$ by the First Terms of Their Asymptotics

A. Yu. Popov[1,2,*] **and T. V. Rodionov**[1,2,**]

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Abstract—Uniform with respect to the parameter $a \in (0, 1)$ estimates of the functions $f_a(x) = \sum_{k=1}^{\infty} k^{-a} \cos kx$ and $g_a(x) = \sum_{k=1}^{\infty} k^{-a} \sin kx$ by the first terms of their asymptotic expansions $F_a(x) = \sin(\pi a/2)\Gamma(1-a)x^{a-1}$ and $G_a(x) = \cos(\pi a/2)\Gamma(1-a)x^{a-1}$ are obtained. Namely, it is proved that the inequalities

$$G_a(x) - \frac{x}{2} < g_a(x) < G_a(x) - \frac{x}{12},$$

$$F_a(x) + \zeta(a) + \frac{\zeta(3)}{4\pi^3}\, x^2 \sin(\pi a/2) < f_a(x) < F_a(x) + \zeta(a) + \frac{1}{18}\, x^2 \sin(\pi a/2)$$

are valid for all $a \in (0, 1)$ and $x \in (0, \pi]$.

It is shown that the estimates are unimprovable in the following sense. In the lower estimate for the sine series, the subtrahend $x/2$ cannot be replaced by kx with any $k < 1/2$: the estimate ceases to be fulfilled for sufficiently small x and the values of a close to 1. In the upper estimate, the subtrahend $x/12$ cannot be replaced by kx with any $k > 1/12$: the estimate ceases to be fulfilled for the values of a and x close to 0. In the lower estimate for the cosine series, the multiplier $\zeta(3)/(4\pi^3)$ of $x^2 \sin(\pi a/2)$ cannot be replaced by any larger number: the estimate ceases to be fulfilled for x and a close to 0. In the upper estimate for the cosine series, the multiplier $1/18$ of $x^2 \sin(\pi a/2)$ can probably be replaced by a smaller number but not by $1/24$: for every $a \in [0.98, 1)$, such an estimate would not hold at the point $x = \pi$ as well as on a certain closed interval $x_0(a) \le x \le \pi$, where $x_0(a) \to 0$ as $a \to 1-$. The obtained results allow us to refine the estimates for the functions f_a and g_a established recently by other authors.

Keywords: special trigonometric series, polylogarithm, periodic zeta function.

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[1]Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991 Russia
[2]Moscow Center of Fundamental and Applied Mathematics, Moscow, 119991 Russia
e-mail: *station@list.ru, **rodionovtv@mail.ru