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ON THE FRACTIONAL NEWTON METHOD WITH CAPUTO DERIVATIVES

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Newton's method is commonly used to solve nonlinear algebraic equations due to its quadratic rate of convergence in the vicinity of the root. Multiple modifications of Newton's method are known, some lead to more stable calculations, although often at the expense of the rate of convergence. Here, derivative in Newton's method is replaced by Caputo fractional derivative, and the goal is to find all the roots, including complex, of nonlinear algebraic equation starting from the same real initial guess by varying the order of fractional derivative. This problem was analyzed by Akgül et al (2019), here some issues with their theoretical analysis and application of the method to the specific example are pointed out. The case of Caputo fractional derivatives of order $(0,1]$ is analyzed. Akgül et al 2019 employ Caputo fractional Taylor's series of Odibat and Shawagfeh, 2007 for theoretical analysis. Specific issues with the paper are the following: 1) In iterative step integration in fractional derivative is done over interval $[\bar{x}, x_k]$, where $\bar{x}$ is the unknown root, and x_k is the approximation of the root on the k-th iteration. 2) Expression for the derivative of fractional Taylor's series is only valid if derivative is evaluated over $[\bar{x}, x_k]$. 3) Expression for the rate of convergence is not correct. 4) In theoretical analysis, left fractional Caputo Taylor series is used, although if $x_{k+1} < \bar{x}$, then right fractional Taylor series should be used. 5) Numerical estimation of the rate of convergence gave value different from predicted by Akgül et al 2019. Plus, not clear over which interval integration was done to generate the numerical results.

Keywords: nonlinear equations, Caputo fractional derivative, Newton's method, convergence.

О дробном методе Ньютона с производными по Капуто. Эмине Челик, Юлонг Ли, Алексей С. Теляковский.

Метод Ньютона часто используется для решения нелинейных алгебраических уравнений, так как он имеет квадратичную скорость сходимости вблизи корня уравнения. Существует много модификаций метода Ньютона, некоторые приводят к более устойчивым вычислениям, хотя при этом может страдать скорость сходимости. В данной работе производная в методе Ньютона заменена нецелочисленной производной по Капуто, и целью является нахождение всех корней, включая комплексные, нелинейного алгебраического уравнения, начиная вычисления из одного и того же вещественного приближения, изменяя только порядок нецелочисленной производной. Эта задача была рассмотрена Акгул в 2019 г. Здесь указаны недочеты теоретического анализа и применения метода к конкретному примеру в упомянутой работе Акгул. Рассмотрен случай нецелочисленных производных по Капуто порядка $(0,1]$. Акгул в работе 2019 использует нецелочисленный Капуто ряд Тейлора в смысле Одибата и Шавагфе 2007 г. Конкретные недочеты следующие: 1) во время итераций интегрирование в нецелочисленной производной проводится по интервалу $[\bar{x}, x_k]$, где $\bar{x}$ — неизвестный корень, а x_k — приближение корня на k-й итерации, 2) выражение для производной нецелочисленного ряда справедливо, только если производная вычислена на интервале $[\bar{x}, x_k]$, 3) выражение для скорости сходимости неверно, 4) во время теоретического анализа используется левый нецелочисленный ряд Тейлора в смысле Капуто, хотя если $x_{k+1} < \bar{x}$, то должен использоваться правый ряд Тейлора, 5) численная оценка скорости сходимости дала значение, отличное от полученного в работе Акгул. Кроме того, не ясно, по какому промежутку производилось интегрирование для получения численных результатов.

Ключевые слова: нелинейные уравнения, дробная производная по Капуто, метод Ньютона, сходимость.

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1. Introduction

In the original paper [1], Akgül et al consider a variation of Newton's method to solve nonlinear algebraic equations $f(x) = 0$, where conventional derivative is replaced by a fractional derivative. In addition, the convergence results based on fractional Taylor series [7] are presented. The discussers would like to comment on authors' results that involve Caputo fractional derivatives only. We discuss some issues with the theoretical analysis of Newton's method in [1].

2. Issues with fractional Newton method

In [1], the following theorem is stated:

Theorem 1 [1, Theorem 2]. *Let the continuous function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has fractional derivatives of order $k\alpha$, for any positive integer k and any α, $0 < \alpha \leq 1$, in the interval D containing the zero $\bar{x}$ of $f(x)$. Let us also suppose that ${}_C D^{\alpha}_{\bar{x}} f(x)$ is continuous and not null in $\bar{x}$. If the initial approximation x_0 is sufficiently close to $\bar{x}$, then the local convergence order of the fractional Newton method of Caputo-type*

$$x_{k+1} = x_k - \Gamma(\alpha+1)\frac{f(x_k)}{{}_C D^{\alpha}_{a} f(x_k)}, \quad k = 0, 1, \dots,$$

is at least 2α, being $0 < \alpha \leq 1$ and the error equation is,

$$e^{\alpha}_{k+1} = \frac{\Gamma(2\alpha+1) - (\Gamma(\alpha+1))^2}{(\Gamma(\alpha+1))^3} C_2 e^{2\alpha}_k + O(e^{3\alpha}_k).$$

Here, ${}_C D^{\alpha}_{a} f(x)$ is the left Caputo fractional derivative of order $0 < \alpha \leq 1$:

$${}_C D^{\alpha}_{a} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_a^x \frac{df(t)}{dt} \frac{dt}{(x-t)^{\alpha}}, & 0 < \alpha \leq 1, \\ \dfrac{df}{dt}, & \alpha = 1. \end{cases}$$

In the proof of this Theorem, the authors present the expression for ${}_C D^{\alpha}_{a} f(x)$, $x > a$, as

$${}_C D^{\alpha}_{a} f(x) = \frac{{}_C D^{\alpha}_{\bar{x}} f(\bar{x})}{\Gamma(\alpha+1)} \Big[\Gamma(\alpha+1) + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} C_2 (x-\bar{x})^{\alpha} + \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} C_3 (x-\bar{x})^{2\alpha}\Big] + O\big((x-\bar{x})^{3\alpha}\big),$$

where

$$C_j = \frac{\Gamma(\alpha+1)}{\Gamma(j\alpha+1)} \frac{{}_C D^{j\alpha}_{\bar{x}} f(\bar{x})}{{}_C D^{\alpha}_{\bar{x}} f(\bar{x})}, \quad \text{for} \ \ j \geq 2.$$

Such representation follows from fractional Taylor expansion for

$$f(x) = \frac{{}_C D^{\alpha}_{\bar{x}} f(\bar{x})}{\Gamma(\alpha+1)} \big[(x-\bar{x})^{\alpha} + C_2 (x-\bar{x})^{2\alpha} + C_3 (x-\bar{x})^{3\alpha}\big] + O\big((x-\bar{x})^{4\alpha}\big),$$

only when $a = \bar{x}$, where $\bar{x}$ is the sought root. The same form for ${}_C D^{\alpha}_{a} f(x)$ is used in the iterative step of this Theorem:

$$x_{k+1} = x_k - \Gamma(\alpha+1)\frac{f(x_k)}{{}_C D^{\alpha}_{a} f(x_k)}, \quad k = 0, 1, \dots.$$

It implies that interval of integration in the above fractional derivative is $[a, x_k] = [\bar{x}, x_k]$, where $\bar{x}$ is the unknown root. So, a is something else here, but $\bar{x}$. While if $a \neq \bar{x}$, then the presented proof of Theorem (Theorem 2 [1]) is not valid.

The discussers applied Newton's algorithm to the first test function

$$f_1(x) = -12.84x^6 - 25.6x^5 + 16.55x^4 - 2.21x^3 + 26.71x^2 - 4.29x - 15.21$$

in [1], when Caputo derivative was evaluated with Matlab code [4] that is based on the algorithms introduced in [2; 5; 6]. This code provides high-accuracy approximation for the Caputo fractional derivatives on an interval $[0, x_k]$. The numerical results of the discussers, given in Table 1, are rather similar to CFN (Caputo fractional Newton) part of Table 1 [1], i. e., the value of the root $\bar{x}$, stopping

Table 1. CFN method results for $f_1(x)$ and initial estimate $x_0 = -2.2$ with 30 points. Matlab code [4] was used to evaluate Caputo fractional derivative

α	$\bar{x}$	$\lvert x_{k+1} - x_k\rvert$	$\lvert f(x_{k+1})\rvert$	k
0.7	−0.62043 + 0.02932i	0.05864	2.99669	500
0.72	−0.59121 + 0.00609i	0.01218	0.57798	500
0.74	−0.5840 + 5e-09i	9.88e-09	4.40e-07	251
0.76	−0.21706 + 0.99911i	8.86e-09	5.99e-07	50
0.78	−0.21706 + 0.99911i	6.87e-09	4.32e-07	37
0.8	−0.21706 + 0.99911i	6.79e-09	3.94e-07	32
0.82	−0.5840 + 4e-09i	9.79e-09	3.18e-07	47
0.84	0.82366 + 0.24769i	3.17e-09	5.21e-08	28
0.86	0.82366 + 0.24769i	3.70e-09	5.39e-08	27
0.88	0.82366 + 0.24769i	8.72e-09	1.11e-07	27
0.9	0.82366 + 0.24769i	5.34e-09	5.75e-08	35
0.92	−2.62298 + 5e-10i	3.28e-09	1.82e-06	32
0.94	−2.62298 + 1e-9i	6.42e-09	2.73e-06	23
0.96	−2.62298 − 1e-9i	9.85e-09	2.88e-06	18
0.98	−2.62298 − 2e-10i	3.53e-09	5.56e-07	15

criteria $|x_{k+1}-x_k|$, $|f(x_{k+1})|$, and the number of iterations. The discussers believe that the fractional Newton's method behaves in many ways similarly to the conventional Newton's method; various modifications often do not change the overall convergence, see e. g., [3]. It means that if $a = 0$, then the fractional Newton's method will converge for some α's for the first test function $f_1(x)$. This observation is also confirmed by equation (1) that follows.

Additionally, the discussers numerically estimated the order of convergence of CFN method for $f_1(x)$, when $\alpha = 0.98$ and $a = 0$. The initial guess was the same as in the original paper, $x_0 = -2.2$. At first, the order of convergence was about 2, which is comparable to the theoretical estimate 2α. While for the later iterations, the order of convergence was about 1. It happens since after few iterations fractional derivative is evaluated over essentially the same interval $[0, \bar{x}]$, and as a result ${}_CD_0^{\alpha} f(x_k) \approx {}_CD_0^{\alpha} f(\bar{x})$. When derivative in conventional Newton's method is replaced by a constant, it leads to linear convergence, see e. g., [3]. The same holds for this version of Newton's method. In our numerical experiment, at first, x_k changes substantially and convergence is quadratic, while later, x_k does not change much and the rate of convergence is linear.

Repeating the authors' calculations, the discussers obtained the following expression for the error term in this Theorem:

$$e_{k+1} = e_k - e_k^{\alpha} + \frac{\Gamma(2\alpha+1) - (\Gamma(\alpha+1))^2}{(\Gamma(\alpha+1))^2} C_2 e_k^{2\alpha} + O(e_k^{3\alpha}). \quad (1)$$

Such form of the error term does not imply that the order of the method is 2α. That is, $e_{k+1} \neq C e_k^{2\alpha}$. Although we can see that the order α is guaranteed by the error estimate, when $a = \bar{x}$. However, the optimal order is still unclear.

Also, for the theoretical analysis of Newton's method the authors rely on the left Caputo fractional Taylor series [7]:

Theorem 2 [1, Theorem 1; 7, Theorem 3]. *Let us suppose that ${}_CD_a^{j\alpha} f(x) \in C([a,b])$ for $j = 1, 2, \ldots, n+1$, where $\alpha \in (0,1]$, then we have*

$$f(x) = \sum_{i=0}^{n} {}_CD_a^{i\alpha} f(a) \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} + {}_CD_a^{(n+1)\alpha} f(\xi) \frac{(x-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)},$$

with $a \le \xi \le x$, for all $x \in (a,b]$, where ${}_CD_a^{n\alpha} = {}_CD_a^{\alpha} \cdot {}_CD_a^{\alpha} \cdots {}_CD_a^{\alpha}$ (n times).

Here, it is assumed that $a \leq \xi \leq x$, and the authors take $a = \bar{x}$, but the Newton iterations can lead to $x_{k+1} < \bar{x}$. Formally speaking, if $x_{k+1} < \bar{x}$, then for the theoretical analysis the right fractional Taylor series at $\bar{x}$ should be considered, see e. g., [8].

Conclusion

To summarize, the authors of [1] accomplish their goal of obtaining various roots, including complex, of equation $f(x) = 0$ by varying the order α of the Caputo fractional derivative for the same real initial guess x_0. At the same time, the presented form of the Newton's method and the proof of convergence are not valid the way they are stated, and should be re-expolored in the future.

REFERENCES

1. Akgül A., Cordero A., Torregrosa J.R. A fractional Newton method with 2αth-order of convergence and its stability. *Appl. Math. Lett.*, 2019, vol. 98, pp. 344–351. doi: 10.1016/J.AML.2019.06.028 .
2. Cao J.X, Li C.P., Chen Y.Q. High-order approximation to Caputo derivatives and Caputo-type advection-diffusion equations (II). *Fract. Calc. Appl. Anal.*, 2015, vol. 18, no. 3, pp. 735–761. doi: 10.1515/fca-2015-0045 .
3. Kincaid D, Cheney W. *Numerical analysis*: *Mathematics of scientific computing,* 3*rd revised edition.* American Mathematical Society, 2002, 788 p. ISBN: 0821847880 .
4. Li C.P. *High-order approximation to Caputo derivatives.* MATLAB Central File Exchange, 2021, https://www.mathworks.com/matlabcentral/fileexchange/53924-high-order-approximation-to-caputo-derivatives (Retrieved May 27, 2021).
5. Li C.P, Wu R.F., Ding H.F. High-order approximation to Caputo derivatives and Caputo-type advection-diffusion equations (I). *Communications in Appl. and Industr. Math.*, 2014, vol. 6, no. 2, e-536, pp. 1–32. doi: 10.1685/journal.caim.536 .
6. Li H., Cao J.X., Li C.P. High-order approximation to Caputo derivatives and Caputo-type advection-diffusion equations (III). *J. of Comp. and Appl. Math.*, 2016, vol. 299, pp. 159–175. doi: 10.1016/j.cam.2015.11.037 .
7. Odibat Z.M., Shawagfeh N.T. Generalized Taylor's formula. *Appl. Math. Comput.*, 2007, vol. 186, no. 1, pp. 286–293. doi: 10.1016/j.amc.2006.07.102 .
8. Olsen J.S., Mortensen J., Telyakovskiy A.S. A two-sided fractional conservation of mass equation. *Advances in Water Resources*, 2016, vol. 91, pp. 117–121. doi: 10.1016/j.advwatres.2016.03.007 .

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