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ON THE PROPERTIES OF THE SET OF TRAJECTORIES OF NONLINEAR CONTROL SYSTEMS WITH INTEGRAL CONSTRAINTS ON THE CONTROL FUNCTIONS

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The control systems described by nonlinear differential equations and integral constraints on the control functions are studied. Admissible control functions are chosen from a closed ball of the space L_p, $p \in (1, \infty]$, with radius r and centered at the origin. It is proved that the set of trajectories of the system is continuous at $p = \infty$ with respect to the Hausdorff pseudometric. It is shown that every trajectory is robust with respect to the fast and full consumption of the remaining control resource which implies that to achieve the desired result, it is advisable to spend the available control resource in small portions. This allows to prove that every trajectory can be approximated by the trajectory, generated by full consumption of the control resource.

Keywords: nonlinear control system, set of trajectories, integral constraint, geometric constraint, Hausdorff continuity, robustness.

Н. Гусейин, А. Гусейин, Х. Г. Гусейнов. О свойствах множества траекторий нелинейной управляемой системы с интегральными ограничениями на управляющие функции.

Исследуются управляемые системы, описываемые нелинейными дифференциальными уравнениями с интегральными ограничениями на управляющие функции. Допустимые управляющие функции выбираются из замкнутого шара пространства L_p, $p \in (1, \infty]$, радиуса r с центром в начале координат. Доказано, что множество траекторий системы непрерывно при $p = \infty$ в псевдометрике Хаусдорфа. Показано, что каждая траектория робастна по отношению к быстрому и полному расходованию оставшегося управляющего ресурса, из чего следует, что для достижения желаемого результата целесообразно расходовать имеющийся ресурс управления малыми порциями. Благодаря этому удается доказать, что каждая траектория может быть аппроксимирована траекторией, соответствующей полному расходованию ресурса управления.

Ключевые слова: нелинейная управляемая система, множество траекторий, интегральное ограничение, геометрическое ограничение, непрерывность по Хаусдорфу, робастность.

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Introduction

The nonlinear control systems described by ordinary differential equations arise in different areas of the theory and applications. One of the important notions of control systems theory is the set of system's trajectories which consists of trajectories generated by all admissible control functions. The set of trajectories includes comprehensive information about system's behaviour and can be used for solution of various type of problems arising in the theory of control systems. For example, the attainable set at a given time defined in the phase-state space and consisting of points to which the system's trajectories arrive at a given time, can be defined as the section of the set of trajectories at that given time. The integral funnel of the system is defined in the extended phase-state space and consists of the graphs of all trajectories. Let us emphasize that the attainable sets and integral funnel are also the adequate tools for visualization of the system's behaviour.

Depending on the character of the control efforts, the control systems are characterised as follows: control systems with geometric constraints on the control functions; control systems with integral constraints on the control functions; and control systems with mixed constraints on the control functions which include both the geometric and the integral constraints on the control functions. Geometric constraints characterize the control efforts which are not exhausted by consumption. The control system with geometric constraints on the control functions is a well investigated chapter of the theory of control systems (see, e.g. [1–6]). These systems are also studied in the framework of differential inclusions theory and positional differential games theory (see, e.g. [7–11]).

If the control resource is exhausted by consumption, say as energy, fuel, finance, food, etc., then integral constraint on the control functions is inevitable. Note that integral boundedness of the control functions does not guarantee their geometric boundedness and therefore in the study of the control systems with integral constraints on the control functions arise the additional and essential difficulties. Different topological properties and approximation methods for construction of the set of trajectories, attainable sets and integral funnels of control systems described by different types of integral and differential equations and integral constraints on the control functions are considered in papers [12–22] (also see references therein).

In this paper, the set of trajectories of a control system described by a nonlinear ordinary differential equation is studied. The admissible control functions are chosen from a closed ball of the space L_p, $p \in (1, \infty]$, centered at the origin with radius r. Dependence of the set of trajectories on p at $p = \infty$ is studied. In paper [13] it is proved that the set of trajectories depends on p continuously if $p \in (1, \infty)$. Note that the aforementioned result can be obtained as a corollary of the main result of the paper [19] where in the general case the continuity of the L_p balls with respect to p on the open interval $(1, \infty)$ is proved. In the presented paper, the Hausdorff continuity of the set of trajectories of the control system at $p = \infty$ is established. Since the L_∞ boundedness is a geometric type constraint, the obtained result asserts that for sufficiently large p, the integral boundedness can be replaced by the geometric one, and vice versa, the norm type geometric boundedness can be replaced by integral constraint with sufficiently large p. For given $\varepsilon > 0$ the lower bound for parameter p guaranteeing the ε-closeness between the sets of trajectories generated by integrally constrained and geometrically constrained control functions is determined.

Another problem discussed in the paper is the robustness of the trajectories with respect to the remaining control resource which is essential for the consumption mode of the control resource in the control process. It is shown that the spending of the control resource with big quants on the domains with sufficiently small Lebesgue measures does not cause significant changes in the trajectories of the system. The similar problem is considered in [18] where the dynamics of the control system is described by Urysohn type integral equation.

The paper is organised as follows: In Section 1, we formulate basic conditions imposed on the equation of the system and used in the subsequent analysis. In Section 2 the Hausdorff continuity of the set of trajectories at $p = \infty$ is established (Theorem 1). The lower bound for parameter p guaranteeing ε-closeness between the set of trajectories generated by integrally constrained and geometrically constrained control functions is determined (the equalities (2.7), (2.13) and (2.22)). In Section 3 the robustness of the trajectories, generated by integrally constrained control functions, with respect to the remaining control resource is discussed. Applying this result, we show that every trajectory of the system can be approximated by the trajectory generated by fill consumption of the control resource (Theorem 2) and thus, it is obtained that the Hausdorff distance between the set of system's trajectories and the set of trajectories generated by full consumption of the control resources is zero (Theorem 3).

1. The System's Description

The nonlinear control system described by the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \tag{1.1}$$

is considered where $x \in \mathbb{R}^n$ is the phase-state vector, $u \in \mathbb{R}^m$ is the control vector, $t \in [t_0, \theta]$ is the time, $T = \theta - t_0$. It is assumed that the function $f(\cdot) : [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies the following conditions.

A. The function $f(\cdot) : [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous.

B. For every bounded set $D \subset [t_0, \theta] \times \mathbb{R}^n$ there exist $\gamma_1 = \gamma_1(D) > 0$, $\gamma_2 = \gamma_2(D) > 0$ and $\gamma_3 = \gamma_3(D)$ such that the inequality

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\| \leq [\gamma_1 + \gamma_2(\|u_1\| + \|u_2\|)] \, \|x_1 - x_2\| + \gamma_3 \|u_1 - u_2\|$$

is satisfied for every $(t, x_1, u_1) \in D \times \mathbb{R}^m$ and $(t, x_2, u_2) \in D \times \mathbb{R}^m$.

C. There exists $c > 0$ such that the inequality

$$\|f(t, x, u)\| \leq c\,(\|x\| + 1)\,(\|u\| + 1)$$

is held for every $(t, x, u) \in [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m$.

If the function $(t, x, u) \to f(t, x, u) : [t_0, \theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous with respect to (x, u), then the conditions **B** and **C** are satisfied.

For given $p \in [1, \infty]$ and $r > 0$ we denote

$$U_{p,r} = \big\{u(\cdot) \in L_p\big([t_0, \theta]; \mathbb{R}^m\big) : \|u(\cdot)\|_p \leq r\big\}$$

where $L_p\left([t_0, \theta]; \mathbb{R}^m\right)$ is the space of Lebesgue measurable functions $u(\cdot) : [t_0, \theta] \to \mathbb{R}^m$ such that $\|u(\cdot)\|_p < \infty$. Here $\|u(\cdot)\|_p = \left(\int_{t_0}^{\theta} \|u(t)\|^p dt\right)^{1/p}$ if $p \in [1, \infty)$ and $\|u(\cdot)\|_\infty = \inf\{\rho > 0 : \|u(t)\| \leq \rho$ for almost all $t \in [t_0, \theta]\}$, $\|\cdot\|$ denotes the Euclidean norm.

$U_{p,r}$ is called the set of admissible control functions and every $u(\cdot) \in U_{p,r}$ is said to be an admissible control function.

Let $u_*(\cdot) \in U_{p,r}$. An absolutely continuous function $x_*(\cdot) : [t_0, \theta] \to \mathbb{R}^n$ satisfying the equation $\dot{x}_*(t) = f\left(t, x_*\left(t\right), u_*(t)\right)$ for almost all $t \in [t_0, \theta]$ and initial condition $x_*(t_0) = x_0$ is said to be a trajectory of the system (1.1) generated by the admissible control function $u_*(\cdot) \in U_{p,r}$. The set of trajectories of the system (1.1) generated by all admissible control functions $u(\cdot) \in U_{p,r}$ is denoted by $X_{p,r}(t_0, x_0)$ and is called briefly the set of trajectories of the system (1.1). For given $t \in [t_0, \theta]$ we set

$$X_{p,r}(t; t_0, x_0) = \big\{x(t) \in \mathbb{R}^n \colon x(\cdot) \in X_{p,r}(t_0, x_0)\big\}. \tag{1.2}$$

The set $X_{p,r}(t; t_0, x_0)$ is said to be the attainable set of the system (1.1) at the time t.

Note that conditions **A**–**C** guarantee the existence, uniqueness and extendability of the solutions up to the time θ for every given $u(\cdot) \in U_{p,r}$ and $p \in [1, \infty]$. Moreover, it is possible to verify that conditions **A**–**C** guarantee that the set of trajectories $X_{p,r}(t_0, x_0)$ is a nonempty bounded and path-connected subset of the space $C\left([t_0, \theta]; \mathbb{R}^n\right)$ for every $p \in [1, \infty]$ and is a precompact subset of the space $C\left([t_0, \theta]; \mathbb{R}^n\right)$ for every $p \in (1, \infty]$ where the symbol $C\left([t_0, \theta]; \mathbb{R}^n\right)$ denotes the space of continuous functions $x(\cdot) : [t_0, \theta] \to \mathbb{R}^n$ with norm $\|x(\cdot)\|_C = \max\{\|x(t)\| : t \in [t_0, \theta]\}$. In general, the set of trajectories $X_{p,r}(t_0, x_0)$ is not a closed set (see, e.g. [7; 16]).

From conditions **A**–**C** it also follows that there exists $\gamma_* > 0$ such that $\|x(\cdot)\|_C \leq \gamma_*$ for every $x(\cdot) \in X_{p,r}(t_0, x_0)$ and $p \in [1, \infty]$. Here and henceforth we will have in mind the cylinder $D_n(\gamma_*) = \{(t, x) \in [t_0, \theta] \times \mathbb{R}^n : \|x\| \leq \gamma_*\}$ as the set D in Condition **B**.

The Hausdorff distance between the bounded sets $\Omega \subset C\left([t_0, \theta]; \mathbb{R}^n\right)$ and $W \subset C\left([t_0, \theta]; \mathbb{R}^n\right)$ is denoted by symbol $h_C(\Omega, W)$. Denote

$$B_C(1) = \big\{x(\cdot) \in C\left([t_0, \theta]; \mathbb{R}^n\right) : \|x(\cdot)\|_C \leq 1\big\}. \tag{1.3}$$

2. Continuity of the Set of Trajectories at $p=\infty$

In Theorem 4.2 of the paper [13] it is proved that the set valued map $p \to X_{p,r}(t_0,x_0)$, $p \in (1,\infty)$, is Hausdorff continuous for every fixed $r \in (0,\infty)$. Note that this theorem can be obtained as corollary of the Theorem 3 of the paper [19] where Hausdorff continuity of the L_p balls is shown. In following, the Hausdorff continuity of the set valued map $p \to X_{p,r}(t_0,x_0)$, $p \in (1,\infty]$, at $p=\infty$ is proved. Denote

$$\beta_* = \begin{cases} \min\{1,\, r,\, r|T-1|\} & \text{if } \ T \neq 1\,, \\ 1 & \text{if } \ T = 1, \end{cases} \tag{2.1}$$

$$T_* = \max\{1,\, T\}\,, \tag{2.2}$$

$$\kappa_* = \gamma_3 \cdot \exp\left[\gamma_1 T + 2\gamma_2 r T_*\right]\,. \tag{2.3}$$

It is obvious that

$$T^{(p-1)/p} \leq T_* \tag{2.4}$$

for every $p \in (1,\infty)$.

At first let us prove that set valued map $p \to X_{p,r}(t_0,x_0)$, $p \in (1,\infty]$, is lower semicontinuous at $p=\infty$.

Proposition 1. *For every $\varepsilon \in (0,\beta_*)$ there exists $P_1(\varepsilon) \geq 1$ such that for each $p > P_1(\varepsilon)$ the inclusion*

$$X_{\infty,r}(t_0,x_0) \subset X_{p,r}(t_0,x_0) + \varepsilon\kappa_* B_C(1)$$

is satisfied where $B_C(1)$ is defined by (1.3).

Proof. Let us choose an arbitrary trajectory $x(\cdot) \in X_{\infty,r}(t_0,x_0)$ generated by the control function $u(\cdot) \in U_{\infty,r}$ and define new function $u_p(\cdot) : [t_0,\theta] \to \mathbb{R}^m$, setting

$$u_p(t) = T^{-1/p}\, u(t)\,, \quad t \in [t_0,\theta]\,. \tag{2.5}$$

where $p>1$, $T=\theta-t_0$.

Since $u(\cdot) \in U_{\infty,r}$, then it is not difficult to verify that $\|u_p(\cdot)\|_p \leq r$ for every $p>1$ which yields that $u_p(\cdot) \in U_{p,r}$. Let the function $x_p(\cdot)\colon [t_0,\theta] \to \mathbb{R}^n$ be the trajectory of the system (1.1) generated by the control function $u_p(\cdot) \in U_{p,r}$. Then, $x_p(\cdot) \in X_{p,r}(t_0,x_0)$.

From inclusion $u(\cdot) \in U_{\infty,r}$ and (2.5) it follows that

$$\|u(\cdot) - u_p(\cdot)\|_1 = \int_{t_0}^{\theta} \left\|u(\tau) - T^{-1/p}\cdot u(\tau)\right\| d\tau \leq rT \cdot \left|1 - T^{-1/p}\right|. \tag{2.6}$$

Let us set

$$P_1(\varepsilon) = \begin{cases} \dfrac{1}{\log_T \frac{rT}{rT-\varepsilon}} & \text{if } \ T > 1\,, \\ \dfrac{1}{\log_T \frac{rT}{rT+\varepsilon}} & \text{if } \ T < 1\,, \\ 1 & \text{if } \ T = 1\,. \end{cases} \tag{2.7}$$

If $T=1$, then from (2.5) it follows that $u_p(t)=u(t)$ for every $t \in [t_0,\theta]$ and $p>1$, which yields that

$$\|u_p(\cdot) - u(\cdot)\|_1 = 0 < \varepsilon \tag{2.8}$$

for every $p > P_1(\varepsilon)$ where $P_1(\varepsilon)=1$.

Let $T > 1$. Since $\varepsilon < \beta_*$, then on behalf of (2.1) we have that $\varepsilon < r(T-1)$, and hence

$$0 < \log_T \frac{rT}{rT-\varepsilon} < 1 .$$

The last inequality and (2.7) yield that $P_1(\varepsilon) > 1$ in the case $T > 1$.

Let us choose an arbitrary $p > P_1(\varepsilon)$. According to (2.7) we have

$$p > \frac{1}{\log_T \frac{rT}{rT-\varepsilon}} ,$$

and consequently

$$rT \cdot \left(1 - T^{-1/p}\right) < \varepsilon .$$

From the last inequality and (2.6) it follows that

$$\|u(\cdot) - u_p(\cdot)\|_1 < \varepsilon \tag{2.9}$$

where $T > 1$, $p > P_1(\varepsilon)$, $u(\cdot) \in U_{\infty,r}$ is arbitrarily chosen, $u_p(\cdot) \in U_{p,r}$ is definded by (2.5).

Now assume that $T < 1$. Since $\varepsilon < \beta_*$, then on behalf of (2.1) we have that $\varepsilon < r(1-T)$, and hence

$$0 < \log_T \frac{rT}{rT+\varepsilon} < 1 .$$

From the last inequality and (2.7) it follows that $P_1(\varepsilon) > 1$ in the case $T < 1$.

Choose an arbitrary $p > P_1(\varepsilon)$. Since $T < 1$ then from (2.7) we have

$$p > \frac{1}{\log_T \frac{rT}{rT+\varepsilon}} ,$$

and consequently

$$rT \cdot \left(T^{-1/p} - 1\right) < \varepsilon .$$

The last inequality and (2.6) imply that

$$\|u(\cdot) - u_p(\cdot)\|_1 < \varepsilon \tag{2.10}$$

where $T < 1$, $p > P_1(\varepsilon)$, $u(\cdot) \in U_{\infty,r}$ is arbitrarily chosen, $u_p(\cdot) \in U_{p,r}$ is defined by (2.5).

Thus, from the inequalities (2.8), (2.9) and (2.10) it follows that the inequality

$$\|u(\cdot) - u_p(\cdot)\|_1 < \varepsilon \tag{2.11}$$

is satisfied for every $p > P_1(\varepsilon)$ where $u(\cdot) \in U_{\infty,r}$ is arbitrarily chosen, the function $u_p(\cdot) \in U_{p,r}$ is defined by (2.5).

Now from condition **B** and (2.11) it follows that

$$\begin{aligned}\|x(t) - x_p(t)\| &\le \int_{t_0}^{t} \left[\gamma_1 + \gamma_2(\|u(\tau)\| + \|u_p(\tau)\|)\right] \|x(\tau) - x_p(\tau)\|\, d\tau + \gamma_3 \int_{t_0}^{t} \|u(\tau) - u_p(\tau)\|\, d\tau \\ &\le \int_{t_0}^{t} \left[\gamma_1 + \gamma_2(\|u(\tau)\| + \|u_p(\tau)\|)\right] \|x(\tau) - x_p(\tau)\|\, d\tau + \gamma_3\varepsilon \qquad (2.12)\end{aligned}$$

for every $t \in [t_0, \theta]$ and $p > P_1(\varepsilon)$.

Taking into consideration (2.2), (2.3), (2.4) and (2.12), the inclusions $u(\cdot) \in U_{\infty,r}$, $u_p(\cdot) \in U_{p,r}$, and applying Gronwall-Bellman inequality and Hölder inequality, we obtain

$$\begin{aligned} \|x(t) - x_p(t)\| &\leq \gamma_3 \varepsilon \cdot \exp\left[\int_{t_0}^{\theta} [\gamma_1 + \gamma_2(\|u(\tau)\| + \|u_p(\tau)\|)] \, d\tau \right] \\ &\leq \gamma_3 \varepsilon \cdot \exp\left[\gamma_1 T + \gamma_2 r T + \gamma_2 r T^{(p-1)/p}\right] \\ &\leq \gamma_3 \varepsilon \cdot \exp\left[\gamma_1 T + \gamma_2 r T_* + \gamma_2 r T_*\right] = \gamma_3 \varepsilon \cdot \exp\left[\gamma_1 T + 2\gamma_2 r T_*\right] = \kappa_* \varepsilon \end{aligned}$$

for every $t \in [t_0, \theta]$, and hence

$$\|x(\cdot) - x_p(\cdot)\|_C \leq \kappa_* \varepsilon \, .$$

Since $p > P_1(\varepsilon)$, $x(\cdot) \in X_{\infty,r}(t_0, x_0)$ are arbitrarily chosen, $x_p(\cdot) \in X_{p,r}(t_0, x_0)$, then the last inequality implies the proof of the proposition. □

The next proposition characterizes upper semicontinuty of the set valued map $p \to X_{p,r}(t_0, x_0)$, $p \in (1, \infty]$, at $p = \infty$.

Proposition 2. *For every $\varepsilon \in (0, \beta_*)$ there exists $P_2(\varepsilon) \geq 1$ such that for each $p > P_2(\varepsilon)$ the inclusion*

$$X_{p,r}(t_0, x_0) \subset X_{\infty,r}(t_0, x_0) + \varepsilon \kappa_* B_C(1)$$

is satisfied where $B_C(1)$ is defined by (1.3).

Proof. For given $\varepsilon > 0$ let us set

$$P_2(\varepsilon) = 1 + \log_{\left(\frac{r}{r + \frac{\varepsilon}{2(1+T)}}\right)} \frac{\varepsilon}{2r(1+T)} \, . \tag{2.13}$$

The inequality $\varepsilon < \beta_*$ yields that $\varepsilon < \min\{1, r\}$. So, from (2.13) we obtain that $P_2(\varepsilon) > 1$. Choose an arbitrary $p > P_2(\varepsilon)$ and trajectory $z(\cdot) \in X_{p,r}(t_0, x_0)$ of the system (1.1) generated by the control function $w(\cdot) \in U_{p,r}$. Define a function $w_*(\cdot) : [t_0, \theta] \to \mathbb{R}^m$, setting

$$w_*(t) = \begin{cases} w(t) & \text{if} \quad \|w(t)\| \leq r + \dfrac{\varepsilon}{2(1+T)} \, , \\ \left(r + \dfrac{\varepsilon}{2(1+T)}\right) \dfrac{w(t)}{\|w(t)\|} & \text{if} \quad \|w(t)\| > r + \dfrac{\varepsilon}{2(1+T)} \, . \end{cases} \tag{2.14}$$

It is obvious that $w_*(\cdot) \in U_{\infty, r + \frac{\varepsilon}{2(1+T)}}$. Denote

$$G_* = \left\{ t \in [t_0, \theta] \colon \|w(t)\| > r + \frac{\varepsilon}{2(1+T)} \right\}.$$

Since $w(\cdot) \in U_{p,r}$, then Tchebyshev's inequality (see, [23, p.82]) yields that

$$\mu(G_*) \leq \frac{r^p}{\left(r + \frac{\varepsilon}{2(1+T)}\right)^p} \tag{2.15}$$

where $\mu(G_*)$ stands for the Lebesgue measure of the set G_*.

From (2.14), (2.15) and Hölder's inequality it follows that

$$\begin{aligned} \|w(\cdot) - w_*(\cdot)\|_1 &= \int_{G_*} \|w(\tau)\| \cdot \left| 1 - \frac{r + \frac{\varepsilon}{2(1+T)}}{\|w(\tau)\|} \right| d\tau \leq \int_{G_*} \|w(\tau)\| \, d\tau \\ &\leq [\mu(G_*)]^{(p-1)/p} \left(\int_{G_*} \|w(s)\|^p \, d\tau \right)^{1/p} \leq r \left(\frac{r}{r + \frac{\varepsilon}{2(1+T)}} \right)^{p-1} . \end{aligned} \tag{2.16}$$

Since $p > P_2(\varepsilon)$, then from (2.13) we have

$$p > 1 + \log_{\left(\frac{r}{r+\frac{\varepsilon}{2(1+T)}}\right)} \frac{\varepsilon}{2r(1+T)},$$

and hence

$$r\left(\frac{r}{r+\frac{\varepsilon}{2(1+T)}}\right)^{p-1} < \frac{\varepsilon}{2(1+T)}.$$

From the last inequality and (2.16) we obtain that

$$\|w(\cdot) - w_*(\cdot)\|_1 \le \frac{\varepsilon}{2(1+T)} \le \frac{\varepsilon}{2} \tag{2.17}$$

where $w_*(\cdot) \in U_{\infty, r+\frac{\varepsilon}{2(1+T)}}$. Since $w_*(\cdot) \in U_{\infty, r+\frac{\varepsilon}{2(1+T)}}$, then there exists $w_0(\cdot) \in U_{\infty,r}$ such that

$$\|w_*(\cdot) - w_0(\cdot)\|_\infty \le \frac{\varepsilon}{2(1+T)},$$

and hence

$$\|w_*(\cdot) - w_0(\cdot)\|_1 \le T\frac{\varepsilon}{2(1+T)} \le \frac{\varepsilon}{2}. \tag{2.18}$$

Inequalities (2.17) and (2.18) imply that

$$\|w(\cdot) - w_0(\cdot)\|_1 \le \|w(\cdot) - w_*(\cdot)\|_1 + \|w_*(\cdot) - w_0(\cdot)\|_1 \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{2.19}$$

where $w_0(\cdot) \in U_{\infty,r}$. Let the function $z_0(\cdot)\colon [t_0,\theta] \to \mathbb{R}^n$ be the trajectory of the system (1.1) generated by the control function $w_0(\cdot) \in U_{\infty,r}$. Then $z_0(\cdot) \in X_{\infty,r}(t_0,x_0)$. From Condition **B** and (2.19) we have

$$\|z(t) - z_0(t)\| \le \int_{t_0}^{t} [\gamma_1 + \gamma_2(\|w(\tau)\| + \|w_0(\tau)\|)]\, \|z(\tau) - z_0(\tau)\|\, d\tau + \gamma_3\varepsilon \tag{2.20}$$

for every $t \in [t_0,\theta]$. Taking into consideration (2.2)–(2.4), the inclusions $w(\cdot) \in U_{p,r}$, $w_0(\cdot) \in U_{\infty,r}$ and applying Gronwall–Bellman and Hölder's inequlities, from (2.20) we obtain

$$\begin{aligned}\|z(t) - z_0(t)\| &\le \gamma_3\varepsilon \cdot \exp\left[\int_{t_0}^{\theta} [\gamma_1 + \gamma_2(\|w(\tau)\| + \|w_0(\tau)\|)]\, d\tau\right] \\ &\le \gamma_3\varepsilon \cdot \exp[\gamma_1 T + \gamma_2 r T_* + \gamma_2 r T_*] = \gamma_3\varepsilon \cdot \exp[\gamma_1 T + 2\gamma_2 r T_*] = \kappa_*\varepsilon\end{aligned}$$

for every $t \in [t_0,\theta]$ and hence

$$\|z(\cdot) - z_0(\cdot)\|_C \le \kappa_*\varepsilon. \tag{2.21}$$

Since $p > P_2(\varepsilon)$, $z(\cdot) \in X_{p,r}(t_0,x_0)$ are arbitrarily chosen, $z_0(\cdot) \in X_{\infty,r}(t_0,x_0)$, then the inequality (2.21) completes the proof of the proposition.

From Proposition 1 and Proposition 2 it follows the validity of the following theorem. Let

$$P_*(\varepsilon) = \max\{P_1(\varepsilon),\, P_2(\varepsilon))\} \tag{2.22}$$

where $P_1(\varepsilon)$ and $P_2(\varepsilon)$ are defined in Proposition 1 and Proposition 2 respectively.

Theorem 1. *For every $\varepsilon \in (0,\beta_*)$ and $p > P_*(\varepsilon)$ the inequality*

$$h_C(X_{p,r}(t_0,x_0), X_{\infty,r}(t_0,x_0) \le \kappa_*\varepsilon$$

is verified where $\kappa_* > 0$ is defined by (2.3), $P_*(\varepsilon)$ is defined by (2.22).

Theorem 1 implies the validity of the following corollaries.

Corollary 1. *For every* $\varepsilon \in (0, \beta_*)$ *and* $p > P_*(\varepsilon)$ *the inequality*

$$h_C(X_{p,r}(t; t_0, x_0), X_{\infty,r}(t; t_0, x_0) \le \kappa_* \varepsilon$$

is verified for each $t \in [t_0, \theta]$ *where the set* $X_{p,r}(t; t_0, x_0)$ *is defined by* (1.2).

Corollary 2. *The set valued map* $p \to X_{p,r}(t_0, x_0)$, $p \in (1, \infty]$, *is Hausdorff continuous at* $p = \infty$ *for every fixed* $r \in (0, \infty)$, *i.e.* $h_C(X_{p,r}(t_0, x_0), X_{\infty,r}(t_0, x_0)) \to 0$ *as* $p \to \infty$ *for every fixed* $r \in (0, \infty)$.

Corollary 3. *The set valued map* $p \to X_{p,r}(t; t_0, x_0)$, $p \in (1, \infty]$, *is Hausdorff continuous at* $p = \infty$ *uniformly with respect to* $t \in [t_0, \theta]$ *for every fixed* $r \in (0, \infty)$, *i.e.*

$$h_C(X_{p,r}(t; t_0, x_0), X_{\infty,r}(t; t_0, x_0)) \to 0$$

as $p \to \infty$ *uniformly with respect to* $t \in [t_0, \theta]$ *for every fixed* $r \in (0, \infty)$.

3. Robustness of the Trajectories

Now let us discuss robustness of the trajectories with respect to the remaining control resource. In this section it will be assumed that $p \in (1, \infty)$. Denote

$$V_{p,r} = \{u(\cdot) \in L_p([t_0, \theta]; \mathbb{R}^m) : \|u(\cdot)\|_p = r\}, \tag{3.1}$$

and let $Z_{p,r}(t_0, x_0)$ be the set of trajectories of the system (1.1) generated by all admissible control functions $v(\cdot) \in V_{p,r}$.

The following theorem characterises the robustness of a system's trajectory with respect to the fast and full consumption of the remaining control resource.

Theorem 2. *Let* $\varepsilon > 0$ *be a given number,* $x(\cdot) \in \mathbf{X}_{p,r}(t_0, x_0)$ *be a trajectory of the system* (1.1) *generated by the admissible control function* $u(\cdot) \in U_{p,r}$, $\|u(\cdot)\|_p = r_1 < r$, $E_* \subset [t_0, \theta]$ *be a Lebesgue measurable set, the control function*

$$v(t) = \begin{cases} u(t) & \text{if } \ t \in [t_0, \theta] \setminus E_* \, , \\ u_*(t) & \text{if } \ t \in E_* \end{cases}$$

be such that $\|v(\cdot)\|_p = r$, $y(\cdot) : [t_0, \theta] \to \mathbb{R}^n$ *be the trajectory of the system* (1.1) *generated by the admissible control functions* $v(\cdot) \in U_{p,r}$. *If*

$$\mu(E_*) \le \left[\frac{\varepsilon}{2r\kappa_*}\right]^{p/(p-1)}, \tag{3.2}$$

then $\|x(\cdot) - y(\cdot)\|_C \le \varepsilon$ *where* $\mu(E_*)$ *denotes the Lebesgue measure of the set* E_*, κ_* *is defined by* (2.3).

Proof. From Condition B, inclusions $u(\cdot) \in U_{p,r}$, $v(\cdot) \in V_{p,r} \subset U_{p,r}$ and Hölder's inequality it follows that

$$\begin{aligned} \|x(t) - y(t)\| &\le \int_{t_0}^{t} [\gamma_1 + \gamma_2(\|u(\tau)\| + \|v(\tau)\|)] \, \|x(\tau) - y(\tau)\| \, d\tau + \gamma_3 \int_{E_*} \|u(\tau) - v(\tau)\| \, d\tau \\ &\le \int_{t_0}^{t} [\gamma_1 + \gamma_2(\|u(\tau)\| + \|v(\tau)\|)] \, \|x(\tau) - y(\tau)\| \, d\tau + 2r\gamma_3[\mu(E_*)]^{(p-1)/p} \end{aligned} \tag{3.3}$$

for every $t \in [t_0, \theta]$. Taking into consideration the inclusions $u(\cdot) \in U_{p,r}$, $v(\cdot) \in V_{p,r} \subset U_{p,r}$, Hölder's inequality, (2.3), (2.4), the inequality (3.2) and applying the Gronwall–Bellman inequality, from (3.3) we obtain

$$\|x(t) - y(t)\| \le 2r\gamma_3[\mu(E_*)]^{(p-1)/p} \cdot \exp\left[\int_{t_0}^{\theta} [\gamma_1 + \gamma_2(\|u(\tau)\| + \|v(\tau)\|)]\, d\tau\right]$$
$$\le 2r\gamma_3[\mu(E_*)]^{(p-1)/p} \cdot \exp\left[\gamma_1 T + 2\gamma_2 r T_*\right] \le 2r\kappa_*[\mu(E_*)]^{(p-1)/p} \le \varepsilon$$

for every $t \in [t_0, \theta]$ and hence

$$\|x(\cdot) - y(\cdot)\|_C \le \varepsilon\,.$$

The proof is completed. □

Theorem 3. *The equality $h_C\left(X_{p,r}(t_0, x_0), Z_{p,r}(t_0, x_0)\right) = 0$ holds.*

Proof. Choose an arbitrary $\zeta > 0$ and trajectory $y(\cdot) \in X_{p,r}(t_0, x_0)$ of the system (1.1) generated by the control function $u(\cdot) \in U_{p,r}$. Assume that $\|u(\cdot)\|_p = r_* < r$. And let the set $A_* \subset [t_0, \theta]$ be such that

$$\mu(A_*) \le \left[\frac{\zeta}{2r\kappa_*}\right]^{p/(p-1)}, \tag{3.4}$$

where κ_* is defined by (2.3). Let

$$\int_{[t_0,\theta]\setminus A_*} \|u(\tau)\|^p\, d\tau = r_1^p\,.$$

It is obvious that, $r_1 \le r_*$. Define new control function $u_*(\cdot) : [t_0, \theta] \to \mathbb{R}^m$, setting

$$u_*(t) = \begin{cases} u(t) & \text{if} \quad t \in [t_0, \theta] \setminus A_*\,, \\ \left[\dfrac{r^p - r_1^p}{T}\right]^{1/p} \cdot b & \text{if} \quad t \in A_* \end{cases}$$

where $b \in S = \{s \in \mathbb{R}^m : \|s\| = 1\}$ is an arbitrarily chosen vector.

It is easy to show that $\|u_*(\cdot)\|_p = r$ and hence $u_*(\cdot) \in V_{p,r}$ where the set of controls $V_{p,r}$ is defined by (3.1). Let $y_*(\cdot) : [t_0, \theta] \to \mathbb{R}^n$ be the trajectory of the system (1.1) generated by the control function $u_*(\cdot)$. Then $y_*(\cdot) \in Z_{p,r}(t_0, x_0)$ and by virtue of (3.4) and of the Theorem 2 we have that

$$\|y(\cdot) - y_*(\cdot)\|_C \le \zeta\,.$$

Since $y(\cdot) \in X_{p,r}(t_0, x_0)$ is an arbitrarily chosen trajectory, $y_*(\cdot) \in Z_{p,r}(t_0, x_0)$, then the last inequality implies that

$$X_{p,r}(t_0, x_0) \subset Z_{p,r}(t_0, x_0) + \zeta B_C(1). \tag{3.5}$$

Taking into consideration that $Z_{p,r}(t_0, x_0) \subset X_{p,r}(t_0, x_0)$, from (3.5) we obtain that

$$h_C(X_{p,r}(t_0, x_0), Z_{p,r}(t_0, x_0)) \le \zeta\,.$$

Since $\zeta > 0$ is a an arbitrarily fixed number, then the last inequality completes the proof. □

Note that Theorem 3 can be used for simplification of the approximate construction methods of the set of trajectories of the system (1.1). According to this theorem, for approximate construction of the set of trajectories, it is enough to use only the control functions from the set $V_{p,r}$.

Corollary 4. *The equality $cl\left(X_{p,r}(t_0, x_0)\right) = cl\left(Z_{p,r}(t_0, x_0)\right)$ is verified, where cl denotes the closure of a set.*

Corollary 5. *The equality $cl\left(X_{p,r}(t; t_0, x_0)\right) = cl\left(Z_{p,r}(t; t_0, x_0)\right)$ is verified for every $t \in [t_0, \theta]$.*

Conclusion

Hausdorff continuity property of the set of trajectories at $p = \infty$ allows to assert that if $p > P_*(\varepsilon)$, then the Hausdorff distance between the set of trajectories generated by the integrally constrained control functions $u(\cdot) \in U_{p,r}$ and the set of trajectories generated by geometrically constrained control functions $u(\cdot) \in U_{\infty,r}$ does not exceed $\kappa_* \varepsilon$ where $P_*(\varepsilon)$ and κ_* have explicit expressions and depend on the system's parameters and upper bound of the control resource. This situation permits to use in applications the geometrically constrained control functions set $U_{\infty,r}$ instead of the set of integrally constrained control functions $U_{p,r}$ and vice versa, if $p > P_*(\varepsilon)$. This means that for $p > P_*(\varepsilon)$ the trajectories generated by the integrally constrained and geometrically constrained control functions have the $\kappa_* \cdot \varepsilon$ close behaviours. The robustness of the trajectories with respect to the remaining control resource means that the consumption of the big amounts of the control resource on the domains with sufficiently small Lebesgue measures will cause insignificant changes in the trajectories of the system and therefore it is useful to spend the control resource in economy mode. This fact says also that if you have unwanted control resource, then spending this resource on the domain with small Lebesgue measure, you will obtain a small deviation for the initial trajectory. Finally, the coincidence of the closure of the set of trajectories with the closure of the set of trajectories obtained by full consumption of the control resource allows in the approximate construction methods to use only the control functions from the set $V_{p,r}$ which significantly reduces the number of the trajectories to be constructed.

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