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THE GIRTHS OF THE CUBIC PANCAKE GRAPHS[1]

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The pancake graphs $P_n, n \geqslant 2$, are Cayley graphs over the symmetric group Sym_n generated by prefix-reversals. There are six generating sets of prefix-reversals of cardinality three which give connected Cayley graphs over the symmetric group known as cubic pancake graphs. In this paper we study the girth of the cubic pancake graphs. It is proved that considered cubic pancake graphs have the girths at most twelve.

Keywords: pancake graph, cubic pancake graph, prefix-reversal, girth.

Е. В. Константинова, Сон Ен Гун. Обхваты кубических блинных графов.

Блинный граф $P_n, n \geqslant 2$, — это граф Кэли над симметрической группой Sym_n, порожденный операцией инверсии префикса. Существует шесть порождающих множеств инверсий префиксов мощности 3, которые приводят к связным графам Кэли над симметрической группой, известным под названием кубических блинных графов. В статье изучается обхват кубических графов Кэли. Доказано, что рассматриваемые кубические блинные графы имеют обхват не больше 12.

Ключевые слова: блинный граф, кубический блинный граф, инверсия префикса, обхват.

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1. Introduction

The pancake graph $P_n = (\mathrm{Sym}_n, PR), n \geqslant 2$, is the Cayley graph over the symmetric group Sym_n of permutations $\pi = [\pi_1\pi_2 \ldots \pi_n]$ written as strings in one-line notation, where $\pi_i = \pi(i)$ for any $1 \leqslant i \leqslant n$, with the generating set $PR = \{r_i \in \mathrm{Sym}_n : 2 \leqslant i \leqslant n\}$ of all prefix-reversals r_i inversing the order of any substring $[1, i], 2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e. $[\pi_1 \ldots \pi_i\pi_{i+1} \ldots \pi_n]r_i = [\pi_i \ldots \pi_1\pi_{i+1} \ldots \pi_n]$. This graph is well known because of the open combinatorial pancake problem of finding its diameter [3].

The graph P_n is a connected vertex-transitive $(n-1)$-regular graph of order $n!$ with no loops and multiple edges. It is almost pancyclic [4; 9] since it contains cycles of length ℓ, $6 \leqslant \ell \leqslant n!$, but doesn't contain cycles of length $3, 4$ or 5. Since the length of the shortest cycle contained in the graph is six, hence we have $g(P_n) = 6$ for any $n \geqslant 3$, where $g(P_n)$ is the girth of P_n. The girths of the burnt pancake graphs over the hyperoctahedral group was considered in [2]. The (burnt) pancake graphs are commonly used in computer science to represent interconnection networks [1; 10; 11].

Importance of fixed-degree pancake graphs, in particular, cubic pancake graphs as models of networks was shown in [1] by D. W. Bass and I. H. Sudborough. The authors have considered cubic pancake graphs as induced subgraphs of the pancake graph P_n. The necessary conditions for a set of three prefix-reversals to generate the symmetric group Sym_n were found. In particular, it was shown that the cubic pancake graphs over the symmetric group Sym_n, $n \geqslant 4$, are connected with the following generating sets:

$$BS_1 = \{r_2, r_{n-1}, r_n\};\ BS_2 = \{r_{n-2}, r_{n-1}, r_n\};\ BS_3 = \{r_3, r_{n-2}, r_n\}, \text{where } n \text{ is even;}$$

[1]This paper is based on the results of the 2021 Conference of International Mathematical Centers "Groups and Graphs, Semigroups and Synchronization".

Table 1: The girths of the cubic pancake graphs $P_n^6,\ 5 \leqslant n \leqslant 19$

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$g(P_n^6)$	6	8	10	12	12	16	16	16	20	20	20	24	24	24	28

$$BS_4 = \{r_3, r_{n-1}, r_n\},\ \text{where } n \text{ is odd};\ BS_5 = \{r_{n-3}, r_{n-1}, r_n\},\ \text{where } n \text{ is odd};$$
$$BS_6 = \{r_{n-3}, r_{n-2}, r_n\} \text{ for any } n \geqslant 5.$$

The set BS_2 is known as 'big-3' flips, and the corresponding cubic pancake graph generated by this set is called as the 'big-3' pancake network [10].

In this paper we study cubic pancake graphs and their girths. Our first result is obtained for the cubic pancake graphs $P_n^i = Cay(\mathrm{Sym}_n, BS_i)$ that are Cayley graphs over the symmetric group Sym_n generated by the sets $BS_i,\ i = 1, \ldots, 5$.

Theorem 1.

$$g(P_n^1) = g(P_n^2) = g(P_n^3) = \begin{cases} 6, & \textit{when } n = 4; \\ 8, & \textit{when } n \geqslant 5; \end{cases} \tag{1.1}$$

$$g(P_n^4) = 8, \quad \textit{when } n \geqslant 5 \textit{ is odd}; \tag{1.2}$$

$$g(P_n^5) = \begin{cases} 8, & \textit{when } n = 5; \\ 12, & \textit{when } n \geqslant 7 \textit{ is odd}. \end{cases} \tag{1.3}$$

The computational results on the girths of the cubic pancake graph $P_n^6 = Cay(\mathrm{Sym}_n, BS_6)$ are presented in Table 1 for $5 \leqslant n \leqslant 19$. It was also computed that $g(P_n^6) = 28$ for any $19 \leqslant n \leqslant 33$. One can conjecture that this is true for any $n \geqslant 19$.

The paper is organized as follows. The proof of Theorem 1 is based on the characterization of small cycles in the pancake graphs. We present preliminary results with main definitions and notation in Section 2, where two main results show that for any $n \geqslant 7$ there are no 10-cycles and 11-cycles in P_n^5. Then we prove Theorem 1 in Section 3.

2. Preliminary results

The first results on a characterization of small cycles in the pancake graph were obtained in [5] where the following cycle representation via a product of generating elements was used. A sequence of prefix-reversals $C_\ell = r_{i_0} \ldots r_{i_{\ell-1}}$, where $2 \leqslant i_j \leqslant n$, and $i_j \neq i_{j+1}$ for any $0 \leqslant j \leqslant \ell - 1$, such that $\pi r_{i_0} \ldots r_{i_{\ell-1}} = \pi$, where $\pi \in \mathrm{Sym}_n$, is called *a form of a cycle C_ℓ of length ℓ*. A cycle C_ℓ of length ℓ is also called an ℓ-cycle, and a vertex of P_n is identified with the permutation which corresponds to this vertex. It is evident that any ℓ-cycle can be presented by $2\,\ell$ forms (not necessarily different) with respect to a vertex and a direction. *The canonical form C_ℓ of an ℓ-cycle* is called a form with a lexicographically maximal sequence of indices $i_0 \ldots i_{\ell-1}$. We shortly write $C_\ell = (r_a r_b)^k$ for a cycle form $C_\ell = r_a r_b \ldots r_a r_b$, where $\ell = 2\,k,\ a \neq b,\ r_a r_b$ appears exactly k times and $\pi\, r_a r_b \ldots r_a r_b = \pi$ for any $\pi \in \mathrm{Sym}_n$. The form $C_\ell = (r_a r_b)^k$ is *canonical* if $a > b$. By using this description, the following results characterizing 6- and 7-cycles were obtained.

Theorem 2 [5, Theorem 1, Lemma 3]. *The pancake graph $P_n, n \geqslant 3$, has $(n!)/6$ independent 6-cycles of the canonical form*

$$C_6 = (r_3\, r_2)^3 \tag{2.1}$$

and $n!(n-3)$ distinct 7-cycles of the canonical form

$$C_7 = r_k\, r_{k-1}\, r_k\, r_{k-1}\, r_{k-2}\, r_k\, r_2, \tag{2.2}$$

where $4 \leqslant k \leqslant n$. Each of the vertices of P_n belongs to exactly one 6-cycle and $7(n-3)$ distinct 7-cycles.

The complete characterization of 8-cycles is given by the following theorem.

Theorem 3 [7, Theorem 1.3]. *Each of vertices of $P_n, n \geqslant 4$, belongs to $N = (n^3 + 12n^2 - 103n + 176)/2$ distinct 8-cycles of the following canonical forms*:

$$C_8^1 = r_k\, r_j\, r_i\, r_j\, r_k\, r_{k-j+i}\, r_i\, r_{k-j+i}, \qquad 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n \quad (2.3)$$

$$C_8^2 = r_k\, r_{k-1}\, r_2\, r_{k-1}\, r_k\, r_2\, r_3\, r_2, \qquad 4 \leqslant k \leqslant n \quad (2.4)$$

$$C_8^3 = r_k\, r_{k-i}\, r_{k-1}\, r_i\, r_k\, r_{k-i}\, r_{k-1}\, r_i, \qquad 2 \leqslant i \leqslant k-2,\ 4 \leqslant k \leqslant n \quad (2.5)$$

$$C_8^4 = r_k\, r_{k-i+1}\, r_k\, r_i\, r_k\, r_{k-i}\, r_{k-1}\, r_{i-1}, \qquad 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n \quad (2.6)$$

$$C_8^5 = r_k\, r_{k-1}\, r_{i-1}\, r_k\, r_{k-i+1}\, r_{k-i}\, r_k\, r_i, \qquad 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n \quad (2.7)$$

$$C_8^6 = r_k\, r_{k-1}\, r_k\, r_{k-i}\, r_{k-i-1}\, r_k\, r_i\, r_{i+1}, \qquad 2 \leqslant i \leqslant k-3,\ 5 \leqslant k \leqslant n \quad (2.8)$$

$$C_8^7 = r_k\, r_{k-j+1}\, r_k\, r_i\, r_k\, r_{k-j+1}\, r_k\, r_i, \qquad 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n \quad (2.9)$$

$$C_8^8 = (r_4\, r_3)^4. \quad (2.10)$$

The complete characterization of 9-cycles in the pancake graphs were obtained in [6].

In general, the complete characterization of small cycles in the pancake graphs presented in [5;6] is based on the hierarchical structure of the pancake graphs. The graph $P_n, n \geqslant 4$, is constructed from n copies of $P_{n-1}(i), 1 \leqslant i \leqslant n$, such that each $P_{n-1}(i)$ has the vertex set

$$V_i = \{[\pi_1 \ldots \pi_{n-1} i],$$

where $\pi_k \in \{1, \ldots, n\} \setminus \{i\} : 1 \leqslant k \leqslant n-1\}$, $|V_i| = (n-1)!$, and the edge set

$$E_i = \{\{[\pi_1 \ldots \pi_{n-1} i], [\pi_1 \ldots \pi_{n-1} i] r_j\} : 2 \leqslant j \leqslant n-1\},$$

where $|E_i| = \dfrac{(n-1)!(n-2)}{2}$. Any two copies $P_{n-1}(i)$ and $P_{n-1}(j), i \neq j$, are connected by $(n-2)!$ edges presented as $\{[i\pi_2 \ldots \pi_{n-1} j], [j\pi_{n-1} \ldots \pi_2 i]\}$, where $[i\pi_2 \ldots \pi_{n-1} j] r_n = [j\pi_{n-1} \ldots \pi_2 i]$. Prefix-reversals r_j, $2 \leqslant j \leqslant n-1$, defines *internal edges* in $P_{n-1}(i), 1 \leqslant i \leqslant n$, and the prefix-reversal r_n defines *external edges* between copies. Copies $P_{n-1}(i)$, or just P_{n-1} when it is not important to specify the last element of permutations belonging to the copy, are also called $(n-1)-$copies.

The hierarchical structure of the pancake graphs is used to prove the following two results.

Theorem 4. *In the cubic pancake graphs $P_n^5 = Cay(\mathrm{Sym}_n, \{r_{n-3}, r_{n-1}, r_n\}), n \geqslant 7$, there are no cycles of length* 10. *For $n = 5$ there are* 10*-cycles of the canonical form $C_{10} = (r_5\, r_4)^5$.*

Proof. Since the cubic pancake graphs $P_n^5, n \geqslant 5$, are induced subgraphs of the pancake graph P_n, then let us consider all possible cases for forming 10-cycles in the pancake graphs with taking into account that the generating set of P_n^5 contains only three elements r_{n-3}, r_{n-1}, r_n, where n is odd. If $n = 5$ then there are cycles of length 10 of the canonical form $C_{10} = (r_5\, r_4)^5$ (see Lemma 1 in [8]). If $n \geqslant 7$ then due to the hierarchical structure of the pancake graph P_n, cycles of length 10 could be formed from paths of length $l,\ 2 \leqslant l \leqslant 8$, belonging to different $(n-1)-$copies of P_n.

Further, we consider all possible options for the distribution of vertices by copies.

Without loss of generality we always put $\tau^1 = I_n = [1\,2\,3\, \ldots\, n-1\, n]$.

Case 1: 10-cycle within P_n has vertices from two copies of P_{n-1}.

Suppose that a sought 10-cycle is formed on vertices from copies $P_{n-1}(i)$ and $P_{n-1}(j)$, where $1 \leqslant i \neq j \leqslant n$. It was shown in [5, Lemma 2] that if two vertices π and τ, belonging to the same $(n-1)-$copy, are at the distance at most two, then their external neighbours $\overline{\pi}$ and $\overline{\tau}$ should belong to distinct $(n-1)-$copies. Hence, a sought cycle cannot occur in situations when its two (three)

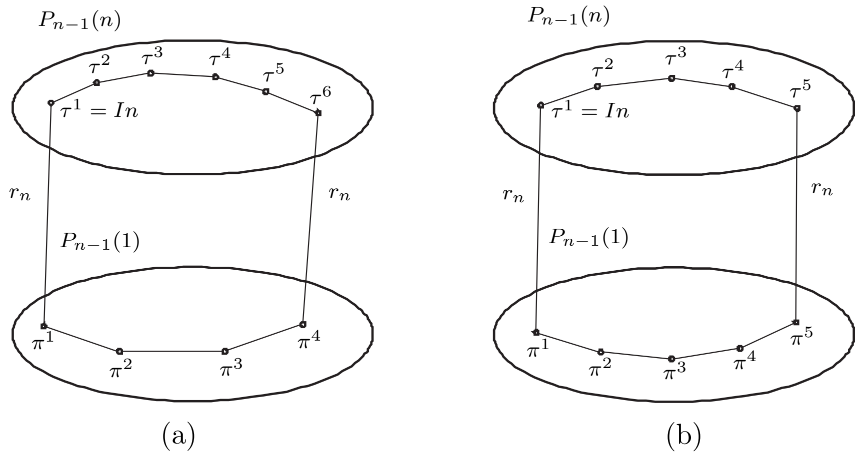


Fig. 1. (a) (4+6)-situation; (b) (5+5)-situation.

vertices belong to one copy and eight (seven) vertices belong to another one. Therefore, such a cycle must have at least four vertices in each of the two copies. Hence, there are two following cases.

Case $(\mathbf{4+6})$. Suppose that four vertices $\pi^1, \pi^2, \pi^3, \pi^4$ of a sought 10-cycle belong to one copy, and other six vertices $\tau^1, \tau^2, \tau^3, \tau^4, \tau^5, \tau^6$ belong to another copy. Let $\pi^1 = \tau^1 r_n$ and $\pi^4 = \tau^6 r_n$. Since $\tau^1 = I_n$ then π^1 and π^4 should belong to $P_{n-1}(1)$. Herewith, the four vertices of $P_{n-1}(1)$ should form a path of length three whose endpoints should be adjacent to vertices from $P_{n-1}(n)$.

Consider all options for passing from τ^1 to τ^6 by internal edges in a copy $P_{n-1}(n)$. Since the generating set of P_n^5 consists of the elements r_{n-1} and r_{n-3} corresponding to internal edges then there are two ways to get paths of length five from τ^1 to τ^6 (see Fig. 1a).

The first path is presented as follows:

$$\tau^1 \, (r_{n-3} \, r_{n-1})^2 \, r_{n-3} = \tau^6 \tag{2.11}$$

such that:

$$\tau^1 = [1\,2\,3\,\ldots\,n{-}1\,n] \xrightarrow{r_{n-3}} [n{-}3\,n{-}4\,\ldots\,2\,1\,n{-}2\,n{-}1\,n] \xrightarrow{r_{n-1}} [n{-}1\,n{-}2\,1\,2\,\ldots\,n{-}4\,n{-}3\,n] \xrightarrow{r_{n-3}}$$

$$[n-5\,n-6\,\ldots\,2\,1\,n-2\,n-1\,n-4\,n-3\,n] \xrightarrow{r_{n-1}} [n-3\,n-4\,n-1\,n-2\,1\,2\,\ldots\,n-6\,n-5\,n] \xrightarrow{r_{n-3}}$$

$$[n-7\,n-8\,\ldots\,2\,1\,n-2\,n-1\,n-4\,n-3\,n-6\,n-5\,n] = \tau^6.$$

Since $\pi^4 = \tau^6 \, r_n$ then we have:

$$\pi^4 = [n\,n-5\,n-6\,n-3\,n-4\,n-1\,n-2\,1\,2\,\ldots\,n-8\,n-7]. \tag{2.12}$$

Note that $\pi^1 = \tau^1 \, r_n = [n\,n-1\,\ldots\,2\,1]$ with $\pi^1_n = 1$ for any $n \geqslant 5$. Hence, we immediately can conclude that π^4 given by (2.12) and π^1 belong to different copies of P_n since $\pi^4_n \neq 1$ for any odd $n \geqslant 7$. For $n = 5$ we get $\pi^4 = [5\,3\,4\,2\,1]$ and there is no path of length three between π^4 and π^1, presented as $r_{n-3}\,r_{n-1}\,r_{n-3}$ or $r_{n-1}\,r_{n-3}\,r_{n-1}$, since $\pi^4\,r_2\,r_4\,r_2 = [4\,2\,5\,3\,1]$ and $\pi^4\,r_4\,r_2\,r_4 = [5\,3\,2\,4\,1]$. This gives a contradiction with an assumption that π^1 and π^4 belong to the same copy $P_{n-1}(1)$. Thus, a sought cycle cannot occur neither in P_n nor in P_n^5.

The second path is presented as follows:

$$\tau^1 \, (r_{n-1} \, r_{n-3})^2 \, r_{n-1} = \tau^6 \tag{2.13}$$

such that

$$\tau^1 = [1\,2\,3\,\ldots\,n-1\,n] \xrightarrow{r_{n-1}} [n-1\,n-2\,\ldots\,2\,1\,n] \xrightarrow{r_{n-3}} [3\,4\,\ldots\,n-2\,n-1\,2\,1\,n] \xrightarrow{r_{n-1}}$$

$$[1\,2\,n-1\,n-2\,\ldots\,4\,3\,n] \xrightarrow{r_{n-3}} [5\,6\,\ldots\,n-2\,n-1\,2\,1\,4\,3\,n] \xrightarrow{r_{n-1}} [3\,4\,1\,2\,n-1\,n-2\,\ldots\,6\,5\,n] = \tau^6,$$

and hence

$$\pi^4 = \tau^6\, r_n = [n\,5\,6\,\ldots\,n-2\,n-1\,2\,1\,4\,3],$$

which means that this permutation belongs to $P_{n-1}(3)$. However, π^1 belongs to $P_{n-1}(1)$ which gives a contradiction again. Thus, a sought 10-cycle cannot occur in this case.

Case $(\mathbf{5+5})$. Suppose that five vertices $\pi^1, \pi^2, \pi^3, \pi^4, \pi^5$ of a sought 10-cycle belong to a copy $P_{n-1}(1)$, and other five vertices $\tau^1, \tau^2, \tau^3, \tau^4, \tau^5$ belong to a copy $P_{n-1}(n)$, where $\tau^1 = I_n$, $\pi^1 = \tau^1 r_n$, and $\pi^5 = \tau^5 r_n$. Then the five vertices of $P_{n-1}(1)$ should form a path of length four whose endpoints should be adjacent to the vertices from $P_{n-1}(n)$ (see Fig. 1b).

There are two ways to get paths of length four from τ^1 to τ^5.

The first way is given as follows:

$$\tau^1\,(r_{n-3}\,r_{n-1})^2 = \tau^5,$$

more precisely, we have:

$$\tau^1 = [1\,2\,3\,\ldots\,n-1\,n] \xrightarrow{r_{n-3}} [n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1\,n] \xrightarrow{r_{n-1}} [n-1\,n-2\,1\,2\,\ldots\,n-4\,n-3\,n] \xrightarrow{r_{n-3}}$$

$$[n-5\,n-6\,\ldots\,2\,1\,n-2\,n-1\,n-4\,n-3\,n] \xrightarrow{r_{n-1}} [n-3\,n-4\,n-1\,n-2\,1\,2\,\ldots\,n-6\,n-5\,n] = \tau^5.$$

The second way is presented as follows:

$$\tau^1\,(r_{n-1}\,r_{n-3})^2 = \tau^5,$$

and we have:

$$\tau^1 = [1\,2\,3\,\ldots\,n-1\,n] \xrightarrow{r_{n-1}} [n-1\,n-2\,\ldots\,2\,1\,n] \xrightarrow{r_{n-3}} [3\,4\,\ldots\,n-2\,n-1\,2\,1\,n] \xrightarrow{r_{n-1}}$$

$$[1\,2\,n-1\,n-2\,\ldots\,4\,3\,n] \xrightarrow{r_{n-3}} [5\,6\,\ldots\,n-2\,n-1\,2\,1\,4\,3\,n] = \tau^5.$$

Since $\pi^5 = \tau^5\, r_n$ then we have either:

$$\pi^5 = [n\,n-5\,n-6\,\ldots\,2\,1\,n-2\,n-1\,n-4\,n-3] \ \text{ or } \ \pi^5 = [n\,3\,4\,1\,2\,n-1\,n-2\,\ldots\,6\,5],$$

which gives a contradiction with an assumption that π^1 and π^5 belong to the copy $P_{n-1}(1)$ for any odd $n \geqslant 5$. Thus, in this case a sought cycle cannot occur in P_n, and hence in P_n^5.

Case 2: 10-cycle within P_n has vertices from three copies of P_{n-1}.

There are four possible situations in this case.

Case $(\mathbf{2+2+6})$. Suppose that two vertices π^1, π^2 of a sought 10-cycle belong to one copy, other two vertices τ^1, τ^2 belong to another copy, and remaining six vertices $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6$ belong to the third copy. Let $\pi^1 = \tau^2 r_n$, $\pi^2 = \gamma^6 r_n$, and $\tau^1 = \gamma^1 r_n = I_n$, then γ^1 and γ^6 should belong to $P_{n-1}(1)$ (see Fig. 2a). It is evident that there are two ways to get paths of length one from τ^1 to τ^2:

$$\tau^1 = [1\,2\,3\,\ldots\,n-1\,n] \xrightarrow{r_{n-3}} [n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1\,n] = \tau^2$$

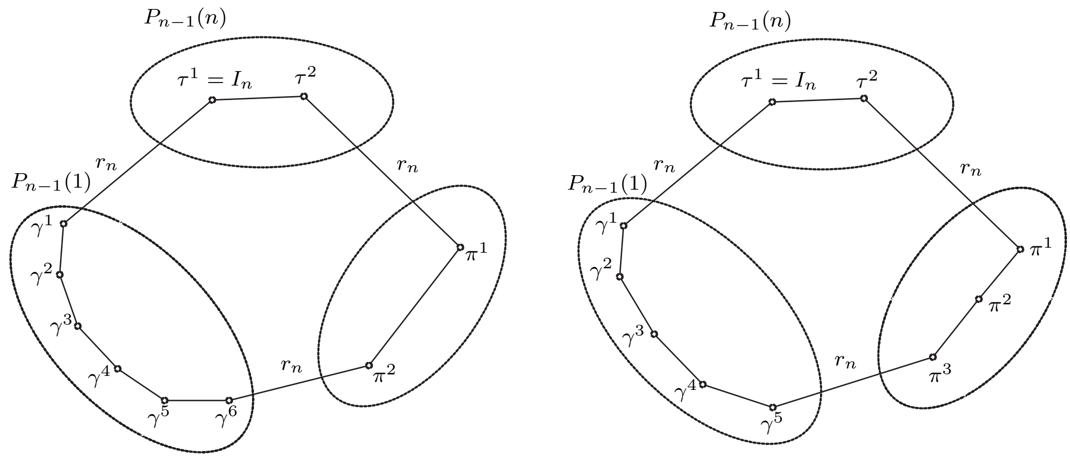
or

$$\tau^1 = [1\,2\,3\,\ldots\,n-1\,n] \xrightarrow{r_{n-1}} [n-1\,n-2\,\ldots\,2\,1\,n] = \tau^2.$$

Since $\pi^1 = \tau^2\, r_n$, we have either

$$\pi^1 = [n\,n-1\,n-2\,1\,2\,\ldots\,n-4\,n-3] \ \text{ or } \ \pi^1 = [n\,1\,2\,\ldots\,n-2\,n-1].$$

Similar, there are two ways to get a path of length one from π^1 to π^2 such that the first way is given by $\pi^1 r_{n-3} = \pi^2$, where we have either $\pi^2 = [n-6\,n-7\,\ldots\,2\,1\,n-2\,n-1\,n\,n-5\,n-4\,n-3]$



(a) (b)

Fig. 2. (a) (2+2+6)-situation; (b) (2+3+5)-situation.

or $\pi^2 = [n-4\; n-5 \ldots 2\; 1\; n\; n-3\; n-2\; n-1]$, and the second way is given by $\pi^1 r_{n-1} = \pi^2$, where we have either $\pi^2 = [n-4\; n-5 \ldots 2\; 1\; n-2\; n-1\; n\; n-3]$ or $\pi^2 = [n-2\; n-3 \ldots 2\; 1\; n\; n-1]$, and since $\gamma^6 = \pi^2 r_n$ we get:

$$\gamma^6 = \begin{cases} [n-3\; n-4\; n-5\; n\; n-1\; n-2\; 1\; 2 \ldots n-7\; n-6] = \gamma^6(A) & \text{or} \\ [n-1\; n-2\; n-3\; n\; 1\; 2 \ldots n-5\; n-4] = \gamma^6(B) & \text{or} \\ [n-3\; n\; n-1\; n-2\; 1\; 2 \ldots n-5\; n-4] = \gamma^6(C) & \text{or} \\ [n-1\; n\; 1\; 2 \ldots n-3\; n-2] = \gamma^6(D). & \end{cases}$$

To get a sought 10-cycle there should be a path of length five between γ^6 and γ^1, where $\gamma^1 = \tau^1 r_n = [n\; n-1 \ldots 2\; 1]$. Let us check this. If $n = 5$ the vertices $\gamma^6(B)$, $\gamma^6(C)$ and $\gamma^1 = [5\,4\,3\,2\,1]$ belong to the copy $P_{n-1}(1)$, and there are two ways to get a path of length five from γ^6 and γ^1. Namely, applying $(r_2\, r_4)^2 r_2$ to γ^6 we have either:

$$\gamma^6(B) = [4\,3\,2\,5\,1] \xrightarrow{r_2} [3\,4\,2\,5\,1] \xrightarrow{r_4} [5\,2\,4\,3\,1] \xrightarrow{r_2} [2\,5\,4\,3\,1] \xrightarrow{r_4} [3\,4\,5\,2\,1] \xrightarrow{r_2} [4\,3\,5\,2\,1] \neq \gamma^1$$

or

$$\gamma^6(C) = [2\,5\,4\,3\,1] \xrightarrow{r_2} [5\,2\,4\,3\,1] \xrightarrow{r_4} [3\,4\,2\,5\,1] \xrightarrow{r_2} [4\,3\,2\,5\,1] \xrightarrow{r_4} [5\,2\,3\,4\,1] \xrightarrow{r_2} [2\,5\,3\,4\,1] \neq \gamma^1,$$

and applying $(r_4 r_2)^2 r_4$ to γ^6 we have either:

$$\gamma^6(B) = [4\,3\,2\,5\,1] \xrightarrow{r_4} [5\,2\,3\,4\,1] \xrightarrow{r_2} [2\,5\,3\,4\,1] \xrightarrow{r_4} [4\,3\,5\,2\,1] \xrightarrow{r_2} [3\,4\,5\,2\,1] \xrightarrow{r_4} [2\,5\,4\,3\,1] \neq \gamma^1$$

or

$$\gamma^6(C) = [2\,5\,4\,3\,1] \xrightarrow{r_4} [3\,4\,5\,2\,1] \xrightarrow{r_2} [4\,3\,5\,2\,1] \xrightarrow{r_4} [2\,5\,3\,4\,1] \xrightarrow{r_2} [5\,2\,3\,4\,1] \xrightarrow{r_4} [4\,3\,2\,5\,1] \neq \gamma^1.$$

Hence, a path of length five does not occur between γ^6 and γ^1.

If $n = 7$ the vertices $\gamma^6(A)$ and $\gamma^1 = [7\,6\,5\,4\,3\,2\,1]$ belong to the copy $P_{n-1}(1)$, and the following two cases are possible:

$$\gamma^6(A)(r_4\, r_6)^2 r_4 = [4\,3\,2\,7\,6\,5\,1](r_4\, r_6)^2 r_4 = [6\,5\,2\,7\,4\,3\,1] \neq \gamma^1$$

or

$$\gamma^6(A)(r_6\, r_4)^2 r_6 = [4\,3\,2\,7\,6\,5\,1](r_6\, r_4)^2 r_6 = [2\,7\,4\,3\,5\,6\,1] \neq \gamma^1.$$

Again, a path of length five does not occur between γ^6 and γ^1.

If $n \geqslant 9$, there is a contradiction with an assumption that γ^1 and γ^6 belong to the copy $P_{n-1}(1)$. Thus, a sought cycle cannot occur in this case.

Case $(\mathbf{2+3+5})$. Suppose that two vertices τ^1, τ^2 of a sought 10-cycle belong to one copy, other three vertices π^1, π^2, π^3 belong to another copy, and remaining five vertices $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$ belong to the third copy. Let $\pi^1 = \tau^2 r_n$, $\pi^3 = \gamma^5 r_n$, and $\tau^1 = \gamma^1 r_n = I_n$, then γ^1 and γ^5 should belong to $P_{n-1}(1)$ (see Fig. 2b). There are two ways to get a path of length two from $\tau^1 = I_n$ to π^1 such that either

$$\tau^1 r_{n-3} r_n = [n\, n-1\, n-2\, 1\, 2\, \ldots\, n-4\, n-3] = \pi^1 \quad \text{or} \quad \tau^1 r_{n-1} r_n = [n\, 1\, 2\, \ldots\, n-2\, n-1] = \pi^1.$$

Similar, there are two ways to get a path of length two from π^1 to π^3. The first one is presented as follows:

$$\pi^1 r_{n-3} r_{n-1} = \pi^3, \tag{2.14}$$

such that either

$$\pi^1 = [n\, n-1\, n-2\, 1\, 2\, \ldots\, n-4\, n-3] \xrightarrow{r_{n-3}} [n-6\, n-7\, \ldots\, 2\, 1\, n-2\, n-1\, n\, n-5\, n-4\, n-3] \xrightarrow{r_{n-1}}$$

$$[n-4\, n-5\, n\, n-1\, n-2\, 1\, 2\, \ldots\, n-7\, n-6\, n-3] = \pi^3$$

or

$$\pi^1 = [n\, 1\, 2\, \ldots\, n-2\, n-1] \xrightarrow{r_{n-3}} [n-4\, n-5\, \ldots\, 2\, 1\, n\, n-3\, n-2\, n-1] \xrightarrow{r_{n-1}}$$

$$[n-2\, n-3\, n\, 1\, 2\, 3\, \ldots\, n-5\, n-4\, n-1] = \pi^3.$$

The second way is presented as follows:

$$\pi^1 r_{n-1} r_{n-3} = \pi^3, \tag{2.15}$$

such that either

$$\pi^1 = [n\, n-1\, n-2\, 1\, 2\, \ldots\, n-4\, n-3] \xrightarrow{r_{n-1}} [n-4\, n-5\, \ldots\, 2\, 1\, n-2\, n-1\, n\, n-3] \xrightarrow{r_{n-3}}$$

$$[n-2\, 1\, 2\, 3\, \ldots\, n-4\, n-1\, n\, n-3] = \pi^3$$

or

$$\pi^1 = [n\, 1\, 2\, \ldots\, n-2\, n-1] \xrightarrow{r_{n-1}} [n-2\, n-3\, \ldots\, 2\, 1\, n\, n-1] \xrightarrow{r_{n-3}} [2\, 3\, \ldots\, n-2\, 1\, n\, n-1] = \pi^3.$$

Since $\pi^3 = \gamma^5\, r_n$, then by (2.14) and (2.15) we obtain:

$$\gamma^5 = \begin{cases} [n-3\, n-6\, n-7\, \ldots 2\, 1\, n-2\, n-1\, n\, n-5\, n-4] = \gamma^5(A) & \text{or} \\ [n-1\, n-4\, \ldots\, 3\, 2\, 1\, n\, n-3\, n-2] = \gamma^5(B) & \text{or} \\ [n-3\, n\, n-1\, n-4\, \ldots\, 3\, 2\, 1\, n-2] = \gamma^5(C) & \text{or} \\ [n-1\, n\ 1\, n-2\, \ldots\, 3\, 2] = \gamma^5(D). \end{cases}$$

To get a sought 10-cycle there should be a path of length four between γ^5 and γ^1, where $\gamma^1 = \tau^1\, r_n = [n\, n-1\, \ldots\, 2\, 1]$. Let us check this. If $n = 5$ the vertices $\gamma^5(A) = I_n\, r_{n-3}\, r_n\, r_{n-3}\, r_{n-1}\, r_n = [2\, 4\, 5\, 3\, 1]$ and $\gamma^1 = [5\, 4\, 3\, 2\, 1]$ belong to the copy $P_{n-1}(1)$, and the following cases are possible:

$$\gamma^5(A)(r_{n-3}\, r_{n-1})^2 = [2\, 4\, 5\, 3\, 1](r_2\, r_4)^2 = [4\, 2\, 3\, 5\, 1] \neq \gamma^1$$

or

$$\gamma^5(A)(r_{n-1}\, r_{n-3})^2 = [2\, 4\, 5\, 3\, 1](r_4\, r_2)^2 = [4\, 2\, 3\, 5\, 1] \neq \gamma^1.$$

Hence, a path of length four does not occur between γ^5 and γ^1. However, let us note that in this case we have a cycle of length eight given by the canonical form $C_8 = (r_4\, r_2)^4$ obtained from (2.9) by putting $k = 4$, $j = 3$, $i = 2$.

If $n \geqslant 7$, there is a contradiction with an assumption that γ^1 and γ^6 belong to the copy $P_{n-1}(1)$. Thus, a sought cycle cannot occur in this case.

Case (2 + 4 + 4). Suppose that two vertices τ^1, τ^2 of a sought 10-cycle belong to one copy, other four vertices $\pi^1, \pi^2, \pi^3, \pi^4$ belong to another copy, and remaining four vertices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$ belong to the third copy (see Fig. 3a). Let $\pi^1 = \tau^1 r_n = [n\, n-1\, n-2\, \ldots\, 2\, 1]$, $\pi^4 = \gamma^4 r_n$, and $\tau^2 = \gamma^1 r_n$. Then both γ^1 and γ^4 should belong to either $P_{n-1}(n-1)$ or $P_{n-1}(n-3)$, since either $\gamma^1 = \tau^1 r_{n-1} r_n$ or $\gamma^1 = \tau^1 r_{n-3} r_n$. More precisely, we have:

$$\gamma^1 = \begin{cases} [n\, 1\, 2\, \ldots\, n-2\, n-1] = \gamma^1(A) & \text{or} \\ [n\, n-1\, n-2\, 1\, 2\, \ldots\, n-4\, n-3] = \gamma^1(B). \end{cases} \tag{2.16}$$

On the other hand, there are two ways to get a path of length three from π^1 to π^4. The first way is presented as follows:

$$\pi^1 r_{n-3} r_{n-1} r_{n-3} = \pi^4 \tag{2.17}$$

such that we have:

$$\pi^1 = [n\, n-1\, n-2\, \ldots\, 5\, 4\, 3\, 2\, 1] \xrightarrow{r_{n-3}} [4\, 5\, \ldots\, n-1\, n\, 3\, 2\, 1] \xrightarrow{r_{n-1}}$$
$$[2\, 3\, n\, n-1 \ldots\, 5\, 4\, 1] \xrightarrow{r_{n-3}} [6\, 7\, \ldots\, n-1\, n\, 3\, 2\, 5\, 4\, 1] = \pi^4.$$

The second way is presented as follows:

$$\pi^1 r_{n-1} r_{n-3} r_{n-1} = \pi^4, \tag{2.18}$$

where we have:

$$\pi^1 = [n\, n-1\, n-2\, \ldots\, 2\, 1] \xrightarrow{r_{n-1}} [2\, 3\, \ldots\, n-2\, n-1\, n\, 1] \xrightarrow{r_{n-3}}$$
$$[n-2\, n-3 \ldots\, 3\, 2\, n-1\, n\, 1] \xrightarrow{r_{n-1}} [n\, n-1\, 2\, 3\, \ldots\, n-3\, n-2\, 1] = \pi^4.$$

Since $\pi^4 = \gamma^4\, r_n$, then by (2.17) and (2.18) we obtain:

$$\gamma^4 = \begin{cases} [1\, 4\, 5\, 2\, 3\, n\, n-1 \ldots\, 7\, 6] = \gamma^4(C) & \text{or} \\ [1\, n-2\, \ldots\, 3\, 2\, n-1\, n] = \gamma^4(D). \end{cases} \tag{2.19}$$

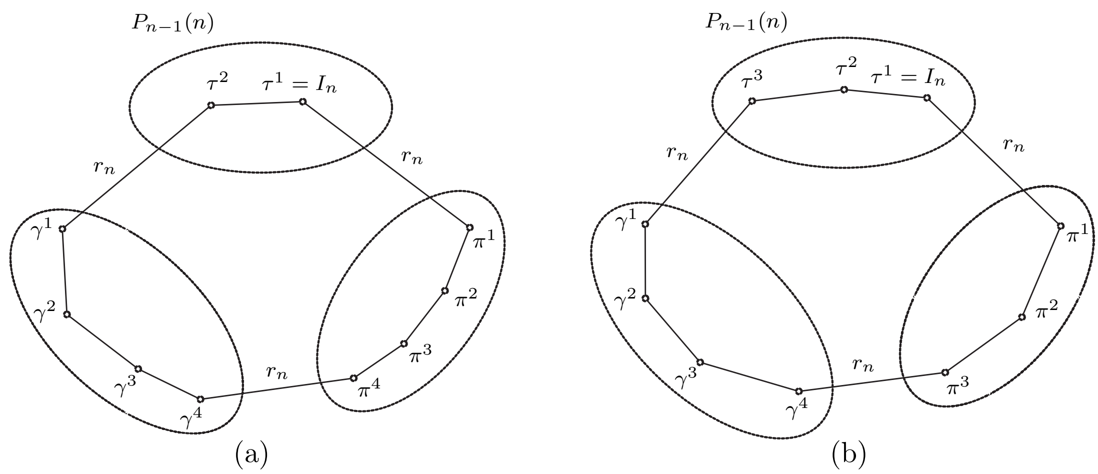


Fig. 3. (a) (2+4+4)-situation; (b) (3+3+4)-situation.

To get a sought 10-cycle there should be a path of length three between γ^1 and γ^4 (see Fig. 3a). Let us check this.

If $n = 5$ or $n \geqslant 11$, there is a contradiction with an assumption that both γ^1 and γ^4 belong to either $P_{n-1}(n-1)$ or $P_{n-1}(n-3)$.

If $n = 7$ then by (2.16) and (2.19) we have $\gamma^1(B) = [7\,1\,2\,3\,4\,5\,6]$ and $\gamma^4(C) = [1\,4\,5\,2\,3\,7\,6]$, but there is no path of length three between them, since we have either

$$\gamma^4(C)\, r_4\, r_6\, r_4 = [4\,1\,3\,7\,5\,2\,6] \neq \gamma^1(B) \ \text{ or } \ \gamma^4(C)\, r_6\, r_4\, r_6 = [1\,4\,7\,3\,2\,5\,6] \neq \gamma^1(B).$$

If $n = 9$ then by (2.16) and (2.19) we have $\gamma^1(A) = [9\,8\,7\,1\,2\,3\,4\,5\,6]$ and $\gamma^4(C) = [1\,4\,5\,2\,3\,9\,8\,7\,6]$, but there is no path of length three between them, since we have either

$$\gamma^4(C)\, r_6\, r_8\, r_6 = [2\,5\,4\,1\,8\,7\,3\,9\,6] \neq \gamma^1(A) \ \text{ or } \ \gamma^4(C)\, r_8\, r_6\, r_8 = [1\,4\,7\,8\,9\,3\,2\,5\,6] \neq \gamma^1(A).$$

Thus, a sought cycle cannot occur in this case.

Case $(\mathbf{3+3+4})$. Suppose that three vertices τ^1, τ^2, τ^3 of a sought 10-cycle belong to one copy, other three vertices π^1, π^2, π^3 belong to another copy, and remaining four vertices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$ belong to the third copy (see Fig. 3b). Let $\pi^1 = \tau^1\, r_n$, $\pi^3 = \gamma^4\, r_n$, and $\tau^3 = \gamma^1\, r_n$. Then both γ^1 and γ^4 should belong to either $P_{n-1}(n-1)$ or $P_{n-1}(3)$, since either $\gamma^1 = \tau^1 r_{n-1} r_{n-3} r_n$ or $\gamma^1 = \tau^1 r_{n-3} r_{n-1} r_n$. More precisely, we have:

$$\gamma^1 = \begin{cases} [n\,1\,2\,n-1\,n-2\,\ldots\,4\,3] = \gamma^1(A) & \text{or} \\ [n\,n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1] = \gamma^1(B). \end{cases} \tag{2.20}$$

On the other hand, since $\pi^1 = \tau^1\, r_n = [n\,n-1\,n-2\,\ldots\,2\,1]$, then by (2.14) and (2.15) there are two ways to get a path of length two from π^1 to π^3 such that either

$$\pi^1 r_{n-3} r_{n-1} = [2\,3\,n\,n-1\ldots\,5\,4\,1] = \pi^3 \ \text{ or } \ \pi^1 r_{n-1} r_{n-3} = [n-2\,n-3\ldots\,3\,2\,n-1\,n\,1] = \pi^3,$$

and since $\pi^3 = \gamma^4\, r_n$, then we have:

$$\gamma^4 = \begin{cases} [1\,4\,5\,\ldots\,n-1\,n\,3\,2] = \gamma^4(C) & \text{or} \\ [1\,n\,n-1\,2\,3\,\ldots\,n-3\,n-2] = \gamma^4(D). \end{cases} \tag{2.21}$$

To get a sought 10-cycle there should be a path of length three between γ^1 and γ^4. Let us check this. If $n = 5$ then by (2.20) and (2.21) we have $\gamma^1(A) = [5\,1\,2\,4\,3]$ and $\gamma^4(D) = [1\,5\,4\,2\,3]$, but there is no path of length three between them, since we have either

$$\gamma^4(D)\, r_2\, r_4\, r_2 = [4\,2\,1\,5\,3] \neq \gamma^1(A) \ \text{ or } \ \gamma^4(D)\, r_4\, r_2\, r_4 = [1\,5\,2\,4\,3] \neq \gamma^1(A).$$

If $n \geqslant 7$ then by (2.20) and (2.21) there is a contradiction with an assumption that both γ^1 and γ^4 belong to either $P_{n-1}(n-1)$ or $P_{n-1}(3)$. Hence, a sought cycle cannot occur in this case.

Case 3: 10-cycle within P_n has vertices from four copies of P_{n-1}.

There are two possible situations in this case.

Case $(\mathbf{2+2+2+4})$. Suppose that two vertices π^1, π^2 of a sought 10-cycle belong to the first copy, two vertices τ^1, τ^2 belong to the second copy, two vertices γ^1, γ^2 belong to the third copy and remaining four vertices $\sigma^1, \sigma^2, \sigma^3, \sigma^4$ belong to the fourth copy (see Fig. 4).

Let $\pi^1 = \tau^2 r_n$, $\pi^2 = \sigma^4 r_n$, $\tau^1 = \gamma^1 r_n$ and $\gamma^2 = \sigma^1 r_n$, then both σ^1 and σ^4 should belong to either $P_{n-1}(4)$ or $P_{n-1}(2)$, since either $\sigma^1 = \tau^1 r_n r_{n-1} r_n$ or $\sigma^1 = \tau^1 r_n r_{n-3} r_n$. More precisely, we have:

$$\sigma^1 = \begin{cases} [1\,n\,n-1\,n-2\,\ldots\,3\,2] = \sigma^1(A) & \text{or} \\ [1\,2\,3\,n\,n-1\,n-2\,\ldots\,6\,5\,4] = \sigma^1(B). \end{cases} \tag{2.22}$$

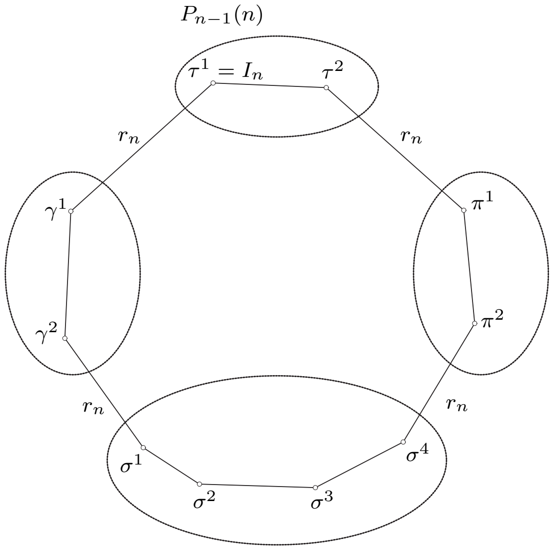


Fig. 4. (2+2+2+4)-situation.

On the other hand, similar to Case $(2+2+6)$ there are four ways to reach σ^4 by a path of length four from τ^1 such that we have:

$$\sigma^4 = \begin{cases} [n-3\, n-4\, n-5\, n\, n-1\, n-2\, 1\, 2\, \ldots\, n-7\, n-6] = \sigma^4(C) & \text{or} \\ [n-1\, n-2\, n-3\, n\, 1\, 2\, \ldots\, n-5\, n-4] = \sigma^4(D) & \text{or} \\ [n-3\, n\, n-1\, n-2\, 1\, 2\, \ldots\, n-5\, n-4] = \sigma^4(E) & \text{or} \\ [n-1\, n\, 1\, 2\, \ldots\, n-3\, n-2] = \sigma^4(F). & \end{cases} \tag{2.23}$$

To get a sought 10-cycle there should be a path of length three between σ^1 and σ^4. Let us check this. If $n = 5$ then by (2.22) we have $\sigma^1(B) = [1\,2\,3\,5\,4]$ and $\sigma^4(C) = I_n\,(r_{n-3}\,r_n)^2 = [2\,1\,3\,5\,4]$, but there is no path of length three between them, since we have either

$$\sigma^4(C)\, r_2\, r_4\, r_2 = [3\,5\,2\,1\,4] \neq \sigma^1(B) \quad \text{or} \quad \sigma^4(C)\, r_4\, r_2\, r_4 = [2\,1\,5\,3\,4] \neq \sigma^1(B).$$

If $n \geqslant 7$ then by (2.22) and (2.23) there is a contradiction with an assumption that both σ^1 and σ^4 belong to either $P_{n-1}(4)$ or $P_{n-1}(2)$. Hence, a sought cycle cannot occur in this case.

Case $(\mathbf{2+2+3+3})$. There are two subcases due to a sequence of vertices from copies forming a cycle: 1) (2,3,3,2); 2) (3,2,3,2) (see Fig. 5)

Subcase 1. Suppose that two vertices τ^1, τ^2 of a sought 10-cycle belong to the first copy, three vertices π^1, π^2, π^3 belong to the second copy, three vertices $\sigma^1, \sigma^2, \sigma^3$ belong to the third copy and remaining two vertices γ^1, γ^2 belong to the fourth copy (see Fig. 5a).

Let $\pi^1 = \tau^2\, r_n$, $\sigma^3 = \pi^3\, r_n$, and $\gamma^1 = \tau^1\, r_n$, $\sigma^1 = \gamma^2\, r_n$. Similar to Case $(2+3+5)$ there are four ways to reach σ^3 by a path of length five from τ^1 such that we have:

$$\sigma^3 = \begin{cases} [n-3\, n-6\, n-7\, \ldots\, 3\, 2\, 1\, n-2\, n-1\, n\, n-5\, n-4] = \sigma^3(A) & \text{or} \\ [n-1\, n-4\, n-5\, \ldots\, 3\, 2\, 1\, n\, n-3\, n-2] = \sigma^3(B) & \text{or} \\ [n-3\, n\, n-1\, n-4\, n-5\, \ldots\, 3\, 2\, 1\, n-2] = \sigma^3(C) & \text{or} \\ [n-1\, n\, 1\, n-2\, n-3\, \ldots\, 3\, 2] = \sigma^3(D). & \end{cases}$$

On the other hand, since $\gamma^1 = \tau^1\, r_n$ and $\tau^1 = I_n$ then $\gamma^1 = [n\, n-1\, n-2\, \ldots\, 2\, 1]$, and there are two ways to get a path of length one from γ^1 to γ^2 such that either

$$\gamma^1 = [n\, n-1\, n-2\, \ldots\, 5\, 4\, 3\, 2\, 1] \xrightarrow{r_{n-3}} [4\, 5\, \ldots\, n-2\, n-1\, n\, 3\, 2\, 1] = \gamma^2$$

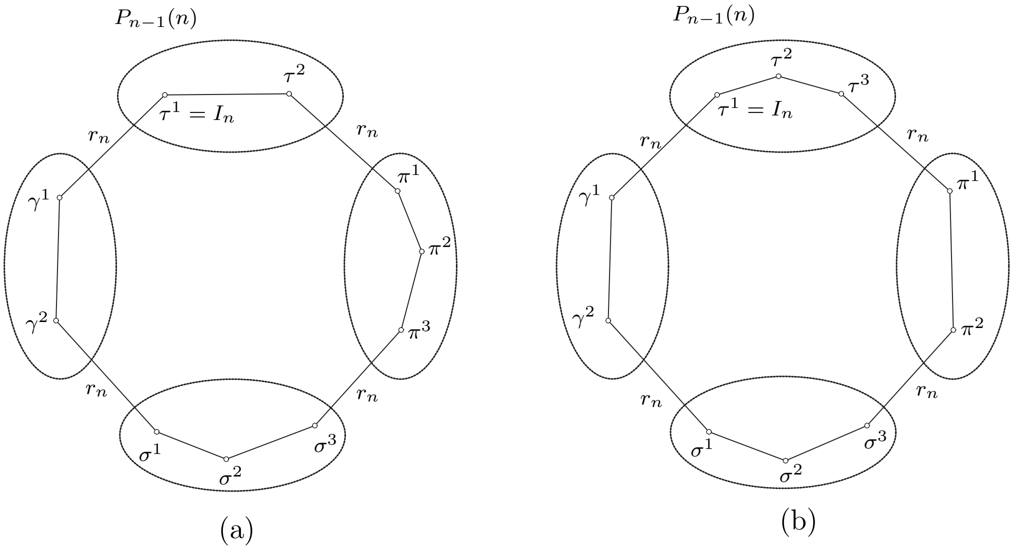


Fig. 5. (2+2+3+3)-situation.

or

$$\gamma^1 = [n\,n-1\,n-2\,\ldots\,3\,2\,1] \xrightarrow{r_{n-1}} [2\,3\,\ldots\,n-2\,n-1\,n\,1] = \gamma^2,$$

and since $\sigma^1 = \gamma^2\, r_n$ then we have either

$$\sigma^1 = [1\,2\,3\,n\,n-1\,n-2\,\ldots\,6\,5\,4] = \sigma^1(E) \quad \text{or} \quad \sigma^1 = [1\,n\,n-1\,n-2\,\ldots\,3\,2] = \sigma^1(F).$$

As one can see, vertices $\sigma^1(F)$ and $\sigma^3(D)$ belong to the same copy $P_{n-1}(2)$. Let us check whether there is a path of length two between these two vertices. Indeed, there are two ways to get a path of length two from $\sigma^1(F)$ to $\sigma^3(D)$. The first way is presented as follows:

$$\sigma^1 r_{n-1} r_{n-3} = \sigma^3, \tag{2.24}$$

where

$$\sigma^1(F) = [1\,n\,n-1\,\ldots\,3\,2] \xrightarrow{r_{n-1}} [3\,4\,\ldots\,n-1\,n\,1\,2] \xrightarrow{r_{n-3}} [n-1\,n-2\,\ldots\,4\,3\,n\,1\,2] \neq \sigma^3(D).$$

The second way is presented as follows:

$$\sigma^1 r_{n-3} r_{n-1} = \sigma^3, \tag{2.25}$$

where

$$\sigma^1(F) = [1\,n\,n-1\,\ldots\,3\,2] \xrightarrow{r_{n-3}} [5\,6\,\ldots\,n-1\,n\,1\,4\,3\,2] \xrightarrow{r_{n-1}} [3\,4\,1\,n\,n-1\,\ldots\,6\,5\,2] \neq \sigma^3(D).$$

Thus, a sought cycle cannot occur in this subcase.

Subcase 2. Suppose that three vertices τ^1, τ^2, τ^3 of a sought 10-cycle belong to the first copy, two vertices π^1, π^2 belong to the second copy, three vertices $\sigma^1, \sigma^2, \sigma^3$ belong to the third copy and remaining two vertices γ^1, γ^2 belong to the fourth copy. Let $\pi^1 = \tau^3 r_n$, $\sigma^3 = \pi^2 r_n$ and $\gamma^1 = \tau^1 r_n = I_n$, $\sigma^1 = \gamma^2 r_n$ (see Fig. 5b). There are two ways to get a path of length two from τ^1 to τ^3. The first way is given as follows:

$$\tau^1 r_{n-3} r_{n-1} = \tau^3, \tag{2.26}$$

where

$$\tau^1 = [1\,2\,3\,\ldots\,n-1\,n] \xrightarrow{r_{n-3}} [n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1\,n] \xrightarrow{r_{n-1}} [n-1\,n-2\,1\,2\,\ldots\,n-4\,n-3\,n] = \tau^3.$$

The second way is given as follows:

$$\tau^1 r_{n-1} r_{n-3} = \tau^3, \tag{2.27}$$

where

$$\tau^1 = [1\,2\,3\,\ldots\,n-1\,n] \xrightarrow{r_{n-1}} [n-1\,n-2\,\ldots\,2\,1\,n] \xrightarrow{r_{n-3}} [3\,4\,\ldots\,n-2\,n-1\,2\,1\,n] = \tau^3.$$

Since $\pi^1 = \tau^3 r_n$, then by (2.26) and (2.27) either $\pi^1 = [n\,n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1]$ or $\pi^1 = [n\,1\,2\,n-1\,n-2\,\ldots\,4\,3]$, and since either $\pi^2 = \pi^1 r_{n-1}$ or $\pi^2 = \pi^1 r_{n-3}$ we have:

$$\pi^2 = \begin{cases} [2\,3\,\ldots\,n-4\,n-3\,n\,1\,n-2\,n-1] = \pi^2(A) & \text{or} \\ [6\,7\,\ldots\,n-2\,n-1\,2\,1\,n\,5\,4\,3] = \pi^2(B) & \text{or} \\ [n-2\,1\,2\,\ldots\,n-4\,n-3\,n\,n-1] = \pi^2(C) & \text{or} \\ [4\,5\,\ldots\,n-2\,n-1\,2\,1\,n\,3] = \pi^2(D), & \end{cases}$$

such that with $\sigma^3 = \pi^2 r_n$ we obtain:

$$\sigma^3 = \begin{cases} [n-1\,n-2\,1\,n\,n-3\,n-4\,\ldots\,3\,2] = \sigma^3(A) & \text{or} \\ [3\,4\,5\,n\,1\,2\,n-1\,n-2\,\ldots\,7\,6] = \sigma^3(B) & \text{or} \\ [n-1\,n\,n-3\,n-4\,\ldots\,2\,1\,n-2] = \sigma^3(C) & \text{or} \\ [3\,n\,1\,2\,n-1\,n-2\,\ldots\,5\,4] = \sigma^3(D). & \end{cases}$$

On the other hand, there are two ways to get a path of length three from τ^1 to σ^1 such that either

$$\sigma^1 = \tau^1\, r_n\, r_{n-3}\, r_n = [1\,2\,3\,n\,\ldots\,6\,5\,4] = \sigma^1(E) \quad \text{or} \quad \sigma^1 = \tau^1\, r_n\, r_{n-1}\, r_n = [1\,n\,n-1\,\ldots\,3\,2] = \sigma^1(F).$$

It is easy to see that vertices $\sigma^1(E)$ and $\sigma^3(D)$ belong to the copy $P_{n-1}(4)$. Let us check whether there is a path of length two between these two vertices. By (2.24) and (2.25), there are two ways to get a path of length two from $\sigma^1(E)$ to $\sigma^3(D)$ such that either

$$\sigma^1(E) = [1\,2\,3\,n\,\ldots\,6\,5\,4] \xrightarrow{r_{n-1}} [5\,6\,\ldots\,n-1\,n\,3\,2\,1\,4] \xrightarrow{r_{n-3}} [3\,n\,n-1\,\ldots\,6\,5\,2\,1\,4] \neq \sigma^3(D),$$

or

$$\sigma^1(E) = [1\,2\,3\,n\,\ldots\,6\,5\,4] \xrightarrow{r_{n-3}} [7\,8\,\ldots\,n-1\,n\,3\,2\,1\,6\,5\,4] \xrightarrow{r_{n-1}} [5\,6\,1\,2\,3\,n\,n-1\,\ldots\,8\,7\,4] \neq \sigma^3(D).$$

Hence, there is no path of length two between these two vertices.

One can also see that vertices $\sigma^1(F)$ and $\sigma^3(A)$ belong to the copy $P_{n-1}(2)$. Let us check whether there is a path of length two between these two vertices. By (2.24) and (2.25), there are two ways to get a path of length two from $\sigma^1(F)$ to $\sigma^3(A)$ such as either

$$\sigma^1(F) = [1\,n\,n-1\,\ldots\,3\,2] \xrightarrow{r_{n-1}} [3\,4\,\ldots\,n-1\,n\,1\,2] \xrightarrow{r_{n-3}} [n-1\,n-2\,\ldots\,4\,3\,n\,1\,2] \neq \sigma^3(A),$$

or

$$\sigma^1(F) = [1\,n\,n-1\,\ldots\,3\,2] \xrightarrow{r_{n-3}} [5\,6\,\ldots\,n-1\,n\,1\,4\,3\,2] \xrightarrow{r_{n-1}} [3\,4\,1\,n\,n-1\,\ldots\,6\,5\,2] \neq \sigma^3(A).$$

Hence, there is no path of length two between these two vertices, and a sought cycle cannot occur in this subcase.

Case 4: 10-cycle within P_n has vertices from five copies of P_{n-1}.

There is the only possible situation if each of five copies has two vertices.

Case $(\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{2})$. Suppose that vertices π^1, π^2 of a sought 10-cycle belong to the first copy, vertices τ^1, τ^2 belong to the second copy, vertices γ^1, γ^2 belong to the third copy, vertices δ^1, δ^2 belong to the fourth copy and vertices σ^1, σ^2 belong to the fifth copy.

Let $\pi^1 = \tau^2 r_n$, $\gamma^1 = \tau^1 r_n$, $\sigma^1 = \gamma^2 r_n$, $\delta^2 = \pi^2 r_n$ and $\delta^1 = \sigma^2 r_n$. Since τ^2 can be reached from τ^1 by either r_{n-1} or r_{n-3}, and the same we have for π^2 and π^1, then there are four ways to get a path of length three from $\tau^1 = I_n$ to π^2 (see Fig. 6) such that:

$$\pi^2 = \begin{cases} [n-6\, n-7\, \ldots\, 2\, 1\, n-2\, n-1\, n\, n-5\, n-4\, n-3] & \text{or} \\ [n-4\, n-5\, \ldots\, 2\, 1\, n\, n-3\, n-2\, n-1] & \text{or} \\ [n-4\, n-5\, \ldots\, 2\, 1\, n-2\, n-1\, n\, n-3] & \text{or} \\ [n-2\, n-3\, \ldots\, 2\, 1\, n\, n-1], \end{cases}$$

and since $\delta^2 = \pi^2 r_n$ we have:

$$\delta^2 = \begin{cases} [n-3\, n-4\, n-5\, n\, n-1\, n-2\, 1\, 2\, \ldots\, n-7\, n-6] & \text{or} \\ [n-1\, n-2\, n-3\, n\, 1\, 2\, \ldots\, n-5\, n-4] & \text{or} \\ [n-3\, n\, n-1\, n-2\, 1\, 2\, \ldots\, n-5\, n-4] & \text{or} \\ [n-1\, n\, 1\, 2\, \ldots\, n-3\, n-2]. \end{cases} \tag{2.28}$$

On the other hand, there are two ways to get a path of length two from τ^1 to γ^2 such that either

$$\tau^1 r_n\, r_{n-3} = [4\, 5\, \ldots\, n-1\, n\, 3\, 2\, 1] = \gamma^2 \quad \text{or} \quad \tau^1 r_n\, r_{n-1} = [2\, 3\, \ldots\, n-1\, n\, 1] = \gamma^2,$$

and since $\sigma^1 = \gamma^2 r_n$ we have:

$$\sigma^1 = \begin{cases} [1\, 2\, 3\, n\, \ldots\, 6\, 5\, 4] & \text{or} \\ [1\, n\, n-1\, \ldots\, 3\, 2], \end{cases}$$

and since either $\sigma^2 = \sigma^1 r_{n-1}$ or $\sigma^2 = \sigma^1 r_{n-3}$ we also have:

$$\sigma^2 = \begin{cases} [7\, 8\, \ldots\, n-1\, n\, 3\, 2\, 1\, 6\, 5\, 4] & \text{or} \\ [5\, 6\, \ldots\, n-1\, n\, 1\, 4\, 3\, 2] & \text{or} \\ [5\, 6\, \ldots\, n-1\, n\, 3\, 2\, 1\, 4] & \text{or} \\ [3\, 4\, \ldots\, n-1\, n\, 1\, 2], \end{cases}$$

which gives us $\delta^1 = \sigma^2 r_n$ as follows:

$$\delta^1 = \begin{cases} [4\, 5\, 6\, 1\, 2\, 3\, n\, n-1\, \ldots\, 8\, 7] & \text{or} \\ [2\, 3\, 4\, 1\, n\, n-1\, \ldots\, 6\, 5] & \text{or} \\ [4\, 1\, 2\, 3\, n\, n-1\, \ldots\, 6\, 5] & \text{or} \\ [2\, 1\, n\, n-1\, \ldots\, 4\, 3]. \end{cases} \tag{2.29}$$

To get a sought 10-cycle, either $\delta^1 = \delta^2 r_{n-3}$ or $\delta^1 = \delta^2 r_{n-1}$ should hold. Let us check this.

If $n = 5$, then by (2.28) and (2.29) we have:

$$\delta^1 = \begin{cases} [4\, 5\, 3\, 1\, 2] = I_n (r_n\, r_{n-3})^2\, r_n & \text{or} \\ [2\, 3\, 4\, 1\, 5] & \text{or} \\ [4\, 1\, 2\, 3\, 5] & \text{or} \\ [2\, 1\, 5\, 4\, 3], \end{cases} \qquad \delta^2 = \begin{cases} [4\, 3\, 2\, 5\, 1] & \text{or} \\ [4\, 5\, 1\, 2\, 3] & \text{or} \\ [2\, 1\, 3\, 5\, 4] & \text{or} \\ [2\, 5\, 4\, 3\, 1]. \end{cases}$$

As one can see, there are two vertices $[2\, 1\, 5\, 4\, 3]$ and $[4\, 5\, 1\, 2\, 3]$ belonging to the same copy, and there is the only way to get a sought 10-cycle containing these vertices by the canonical form $C_{10} = (r_5\, r_4)^5$.

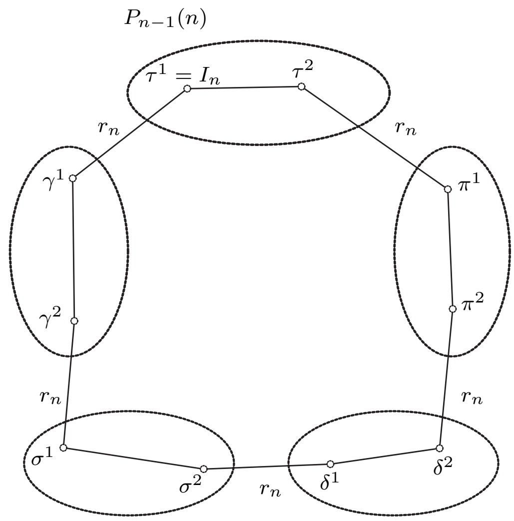


Fig. 6. (2+2+2+2+2)-situation.

If $n = 7$, then by (2.28) and (2.29) we have:

$$\delta^1 = \begin{cases} [4\,5\,6\,1\,2\,3\,7] & \text{or} \\ [2\,3\,4\,1\,7\,6\,5] & \text{or} \\ [4\,1\,2\,3\,7\,6\,5] & \text{or} \\ [2\,1\,7\,6\,5\,4\,3], \end{cases} \qquad \delta^2 = \begin{cases} [4\,3\,2\,7\,6\,5\,1] & \text{or} \\ [6\,5\,4\,7\,1\,2\,3] & \text{or} \\ [4\,7\,6\,5\,1\,2\,3] & \text{or} \\ [6\,7\,1\,2\,3\,4\,5]. \end{cases}$$

It is evident that neither $[6\,5\,4\,7\,1\,2\,3]$ nor $[4\,7\,6\,5\,1\,2\,3]$ could not be reached from $[2\,1\,7\,6\,5\,4\,3]$ neither by r_4 nor by r_6. Thus, a sought 10-cycle can not occur in this case. By using similar arguments, one can check that for $n = 9, 11, 13$ there is no 10-cycle in the graph. For any odd $n \geqslant 15$, there is a contradiction with an assumption that both δ^1 and δ^2 belong to the same copy.

This complete the proof since all possible cases are considered. A 10-cycle in the graph P_n^5 occurs only in the case when $n = 5$ and with the canonical form $C_{10} = (r_5\, r_4)^5$. $\square$

Theorem 5. *In the cubic pancake graphs $P_n^5, n \geqslant 5$, there are no cycles of length* 11.

Proof. To prove this theorem, we use the same arguments as we used to prove Theorem 4. Namely, we consider all possible cases for forming 11-cycles in the pancake graphs P_n with taking into account that the generating set of P_n^5 contains only three elements r_{n-3}, r_{n-1}, r_n, where n is odd. Due to the hierarchical structure of P_n, cycles of length 11 could be formed from paths of length l, $2 \leqslant l \leqslant 9$, belonging to different $(n-1)-$copies of P_n. Further, we consider all possible options for the distribution of vertices by copies.

Within the proof without loss of generality we always put $\tau^1 = I_n = [1\,2\,3\,\ldots\,n-1\,n]$.

Case 1: 11-cycle within P_n has vertices from two copies of P_{n-1}.

Suppose that a sought 11-cycle is formed on vertices from two different copies of P_{n-1}. By [5, Lemma 2], such a cycle cannot occur if its two (three) vertices belong to one copy and nine (eight) vertices belong to another one. Therefore, a sought cycle must have at least four vertices in each of the two copies. Hence, there are two following cases.

Case $(4+7)$. Suppose that four vertices $\pi^1, \pi^2, \pi^3, \pi^4$ of a sought 11-cycle belong to one copy, and other seven vertices $\tau^1, \tau^2, \tau^3, \tau^4, \tau^5, \tau^6, \tau^7$ belong to another copy. Let $\pi^1 = \tau^1 r_n$, $\pi^4 = \tau^7 r_n$, then π^1 and π^4 should belong to $P_{n-1}(1)$. Herewith, the four vertices of $P_{n-1}(1)$ should form a path of length three whose endpoints should be adjacent to vertices from $P_{n-1}(n)$. On the other hand, it is obvious that there are only two ways to get path of length six from τ^1 to τ^7, namely, either:

$$\tau^1\,(r_{n-3}\,r_{n-1})^3 = [n-5\; n-6\; n-3\; n-4\; n-1\; n-2\; 1\; 2\; \ldots\; n-8\; n-7\; n] = \tau^7$$

or

$$\tau^1\,(r_{n-1}\,r_{n-3})^3 = [7\,8\,\ldots\,n-2\,n-1\,2\,1\,4\,3\,6\,5\,n] = \tau^7.$$

Since $\pi^4 = \tau^7\,r_n$ then we have either:

$$\pi^4 = [n\,n-7\,n-8\ldots\,2\,1\,n-2\,n-1\,n-4\,n-3\,n-6\,n-5]$$

or

$$\pi^4 = [n\,5\,6\,3\,4\,1\,2\,n-1\,n-2\,\ldots\,8\,7].$$

Note that $\pi^1 = \tau^1\,r_n = [n\,n-1\,\ldots\,2\,1]$ with $\pi^1_n = 1$ for any $n \geqslant 5$. Hence, we immediately can conclude that π^4 and π^1 belong to different copies of P_n since $\pi^4_n \neq 1$ for any odd $n \geqslant 5$. This gives a contradiction with an assumption that π^1 and π^4 belong to the same copy $P_{n-1}(1)$. Thus, a sought 11-cycle cannot occur in this case.

Case $(\mathbf{5}+\mathbf{6})$. Suppose that five vertices $\pi^1,\pi^2,\pi^3,\pi^4,\pi^5$ of a sought 11-cycle belong to copy $P_{n-1}(1)$, and other six vertices $\tau^1,\tau^2,\tau^3,\tau^4,\tau^5,\tau^6$ belong to copy $P_{n-1}(n)$, where $\pi^1 = \tau^1 r_n$, and $\pi^5 = \tau^6 r_n$. Herewith, the five vertices of $P_{n-1}(1)$ should form a path of length four whose endpoints should be adjacent to vertices from $P_{n-1}(n)$. By (2.11) and (2.13), there are two ways to get paths of length five from τ^1 to τ^6. Moreover, since $\pi^5 = \tau^6\,r_n$ then we have either:

$$\pi^5 = [n\,n-5\,n-6\,n-3\,n-4\,n-1\,n-2\,1\,2\,\ldots\,n-8\,n-7]$$

or

$$\pi^5 = [n\,5\,6\,\ldots\,n-2\,n-1\,2\,1\,4\,3].$$

Since $\pi^1 = \tau^1\,r_n = [n\,n-1\,\ldots\,2\,1]$ with $\pi^1_n = 1$ for any $n \geqslant 5$, then we immediately can conclude that π^5 and π^1 belong to different copies of P_n since $\pi^5_n \neq 1$ for any odd $n \geqslant 5$. This gives a contradiction with an assumption that π^1 and π^5 belong to the same copy $P_{n-1}(1)$. Thus, a sought 11-cycle cannot occur in this case.

Case 2: 11-cycle within P_n has vertices from three copies of P_{n-1}.

There are five different situations in this case.

Case $(\mathbf{2}+\mathbf{2}+\mathbf{7})$. Suppose that two vertices π^1,π^2 of a sought 11-cycle belong to one copy, other two vertices τ^1,τ^2 belong to another copy, and remaining seven vertices $\gamma^1,\gamma^2,\gamma^3,\gamma^4,\gamma^5,\gamma^6,\gamma^7$ belong to the third copy. Let $\pi^1 = \tau^2 r_n$, $\pi^2 = \gamma^7 r_n$, $\tau^1 = \gamma^1 r_n$, then γ^1 and γ^7 should belong to $P_{n-1}(1)$. Using similar reasoning shown in the proof of Theorem 4, Case (2+2+6), one can conclude that there are four ways to reach γ^7 by a path of length four from $\tau^1 = I_n$ such that we have:

$$\gamma^7 = \begin{cases} [n-3\,n-4\,n-5\,n\,n-1\,n-2\,1\,2\,\ldots\,n-7\,n-6] = \gamma^7(A) & \text{or}\\ [n-1\,n-2\,n-3\,n\,1\,2\,\ldots\,n-5\,n-4] = \gamma^7(B) & \text{or}\\ [n-3\,n\,n-1\,n-2\,1\,2\,\ldots\,n-5\,n-4] = \gamma^7(C) & \text{or}\\ [n-1\,n\,1\,2\,\ldots\,n-3\,n-2] = \gamma^7(D). \end{cases}$$

To get a sought 11-cycle there should be a path of length six between γ^7 and γ^1, where $\gamma^1 = \tau^1\,r_n = [n\,n-1\,\ldots\,2\,1]$. Let us check this. If $n=5$ then the vertices $\gamma^7(B) = [4\,3\,2\,5\,1]$, $\gamma^7(C) = [2\,5\,4\,3\,1]$ and $\gamma^1 = [5\,4\,3\,2\,1]$ belong to the copy $P_{n-1}(1)$, and the following cases are possible:

$$\gamma^7(B)(r_2 r_4)^3 = [2\,5\,3\,4\,1] \neq \gamma^1 \;\text{ or }\; \gamma^7(C)(r_2 r_4)^3 = [4\,3\,5\,2\,1] \neq \gamma^1,$$

and

$$\gamma^7(B)(r_4 r_2)^3 = [5\,2\,4\,3\,1] \neq \gamma^1 \;\text{ or }\; \gamma^7(C)(r_4 r_2)^3 = [3\,4\,2\,5\,1] \neq \gamma^1.$$

Hence, a path of length five does not occur between γ^7 and γ^1.

If $n = 7$ the vertices $\gamma^7(A) = [4\,3\,2\,7\,6\,5\,1]$ and $\gamma^1 = [7\,6\,5\,4\,3\,2\,1]$ belong to the copy $P_{n-1}(1)$, and the following two cases are possible:

$$\gamma^7(A)(r_4\,r_6)^3 = [3\,4\,7\,2\,5\,6\,1] \neq \gamma^1 \;\text{ or }\; \gamma^7(A)(r_6\,r_4)^3 = [4\,3\,7\,2\,5\,6\,1] \neq \gamma^1.$$

Again, a path of length five does not occur between γ^7 and γ^1.

If $n \geqslant 9$, there is a contradiction with an assumption that γ^1 and γ^7 belong to the copy $P_{n-1}(1)$. Thus, a sought cycle cannot occur in this case.

Case $(\mathbf{2+3+6})$. Suppose that two vertices τ^1, τ^2 of a sought 11-cycle belong to the first copy, other three vertices π^1, π^2, π^3 belong to the second copy, and remaining six vertices $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6$ belong to the third copy. Let $\pi^1 = \tau^2 r_n$, $\pi^3 = \gamma^6 r_n$, $\tau^1 = \gamma^1 r_n$, then γ^1 and γ^6 should belong to $P_{n-1}(1)$. Taking into account similar reasoning used in the proof of Theorem 4, Case $(2+3+5)$, one can conclude that there are four ways to reach γ^6 by a path of length five from $\tau^1 = I_n$ such that we have:

$$\gamma^6 = \begin{cases} [n-3\; n-6\; n-7\; \ldots\; 2\,1\; n-2\; n-1\; n\; n-5\; n-4] = \gamma^6(A) & \text{or} \\ [n-1\; n-4\; \ldots\; 3\,2\,1\; n\; n-3\; n-2] = \gamma^6(B) & \text{or} \\ [n-3\; n\; n-1\; n-4\; \ldots\; 3\,2\,1\; n-2] = \gamma^6(C) & \text{or} \\ [n-1\; n\; 1\; n-2\; \ldots\; 3\,2] = \gamma^6(D). \end{cases}$$

To get a sought 11-cycle there should be a path of length five between γ^6 and γ^1, where $\gamma^1 = \tau^1\, r_n = [n\; n-1\; \ldots\; 2\,1]$. Let us check this. If $n = 5$ the vertices $\gamma^6(A) = [2\,4\,5\,3\,1]$ and $\gamma^1 = [5\,4\,3\,2\,1]$ belong to the copy $P_{n-1}(1)$, and the following cases are possible:

$$\gamma^6(A)(r_{n-3}\,r_{n-1})^2\, r_{n-3} = [2\,4\,5\,3\,1](r_2\,r_4)^2 r_2 = [2\,4\,3\,5\,1] \neq \gamma^1$$

or

$$\gamma^6(A)(r_{n-1}\,r_{n-3})^2\, r_{n-1} = [2\,4\,5\,3\,1](r_4\,r_2)^2 r_4 = [5\,3\,2\,4\,1] \neq \gamma^1.$$

Hence, a path of length five does not occur between γ^6 and γ^1.

If $n \geqslant 7$, there is a contradiction with an assumption that γ^1 and γ^6 belong to the copy $P_{n-1}(1)$. Thus, a sought cycle cannot occur in this case.

Case $(\mathbf{2+4+5})$. Suppose that two vertices τ^1, τ^2 of a sought 11-cycle belong to the first copy, other four vertices $\pi^1, \pi^2, \pi^3, \pi^4$ belong to the second copy, and remaining five vertices $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$ belong to the third copy. Let $\pi^1 = \tau^1 r_n$, $\pi^4 = \gamma^5 r_n$, $\tau^2 = \gamma^1 r_n$, then γ^1 and γ^5 should belong to either $P_{n-1}(n-1)$ or $P_{n-1}(n-3)$, since either $\gamma^1 = \tau^1 r_{n-1} r_n$ or $\gamma^1 = \tau^1 r_{n-3} r_n$, where:

$$\gamma^1 = \begin{cases} [n\,1\,2\; \ldots\; n-2\; n-1] = \gamma^1(A) & \text{or} \\ [n\; n-1\; n-2\,1\,2\; \ldots\; n-4\; n-3] = \gamma^1(B). \end{cases} \tag{2.30}$$

By the same reasoning used in the proof of Theorem 4, Case $(2+4+4)$, one can see that there are two ways to reach γ^5 by a path of length five from $\tau^1 = I_n$ such that we have:

$$\gamma^5 = \begin{cases} [1\,4\,5\,2\,3\; n\; n-1 \ldots\; 7\,6] = \gamma^5(C) & \text{or} \\ [1\; n-2\; \ldots\; 3\,2\; n-1\; n] = \gamma^5(D). \end{cases} \tag{2.31}$$

To get a sought 11-cycle there should be a path of length four between γ^1 and γ^5. Let us check this. If $n = 5$ or $n \geqslant 11$, there is a contradiction with an assumption that both γ^1 and γ^5 belong to either $P_{n-1}(n-1)$ or $P_{n-1}(n-3)$. If $n = 7$ then by (2.30) and (2.31) we have $\gamma^1(A) = [7\,1\,2\,3\,4\,5\,6]$ and $\gamma^5(C) = [1\,4\,5\,2\,3\,7\,6]$, but there is no path of length four between them, since we have either

$$\gamma^5(C)(r_4\,r_6)^2 = [2\,5\,7\,3\,1\,4\,6] \neq \gamma^1(A) \;\text{ or }\; \gamma^5(C)(r_6\,r_4)^2 = [3\,7\,4\,1\,2\,5\,6] \neq \gamma^1(A).$$

If $n = 9$ then by (2.30) and (2.31) we have $\gamma^1(B) = [9\,8\,7\,1\,2\,3\,4\,5\,6]$ and $\gamma^5(C) = [1\,4\,5\,2\,3\,9\,8\,7\,6]$, but there is no path of length four between them, since we have either

$$\gamma^5(C)(r_6\,r_8)^2 = [9\,3\,7\,8\,1\,4\,5\,2\,6] \neq \gamma^1(A) \ \text{ or } \ \gamma^5(C)(r_8\,r_6)^2 = [3\,9\,8\,7\,1\,4\,2\,5\,6] \neq \gamma^1(A).$$

Thus, a sought cycle cannot occur in this case.

Case (3 + 3 + 5). Suppose that three vertices τ^1, τ^2, τ^3 of a sought 11-cycle belong to the first copy, other three vertices π^1, π^2, π^3 belong to the second copy, and remaining five vertices $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$ belong to the third copy. Let $\pi^1 = \tau^1 r_n = [n\,n-1\,\ldots\,2\,1]$, $\pi^3 = \gamma^5 r_n$, $\tau^3 = \gamma^1 r_n$, then γ^1 and γ^5 should belong to either $P_{n-1}(n-1)$ or $P_{n-1}(3)$, since either $\gamma^1 = \tau^1\,r_{n-3}\,r_{n-1}\,r_n$ or $\gamma^1 = \tau^1\,r_{n-1}\,r_{n-3}\,r_n$, where

$$\gamma^1 = \begin{cases} [n\,n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1] = \gamma^1(A) & \text{or} \\ [n\,1\,2\,n-1\,n-2\,\ldots\,4\,3] = \gamma^1(B). \end{cases} \tag{2.32}$$

Taking into account the same arguments as we used in the proof of Theorem 4, Case $(3 + 3 + 4)$, one can conclude that there are two ways to reach γ^5 by a path of length five from $\tau^1 = I_n$ such that we have:

$$\gamma^5 = \begin{cases} [1\,4\,5\,\ldots\,n-1\,n\,3\,2] = \gamma^5(C) & \text{or} \\ [1\,n\,n-1\,2\,3\,\ldots\,n-3\,n-2] = \gamma^5(D). \end{cases} \tag{2.33}$$

To get a sought 11-cycle there should be a path of length four between γ^1 and γ^5. Let us check this. If $n = 5$ then by (2.32) and (2.33) we have $\gamma^1(B) = [5\,1\,2\,4\,3]$ and $\gamma^5(D) = [1\,5\,4\,2\,3]$, but there is no path of length four between them, since we have either

$$\gamma^5(D)(r_2\,r_4)^2 = [5\,1\,2\,4\,3] \neq \gamma^1(B) \ \text{ or } \ \gamma^5(D)(r_4\,r_2)^2 = [5\,1\,2\,4\,3] \neq \gamma^1(B).$$

Hence, a path of length four does not occur between γ^5 and γ^1. However, let us note that in this case we have a cycle of length eight given by the canonical form $C_8 = (r_4\,r_2)^4$ obtained from (2.9) by putting $k = 4$, $j = 3$, $i = 2$.

If $n \geqslant 7$ then by (2.32) and (2.33) there is a contradiction with an assumption that both γ^1 and γ^5 belong to either $P_{n-1}(n-1)$ or $P_{n-1}(3)$. Hence, a sought cycle cannot occur in this case.

Case (3 + 4 + 4). Suppose that three vertices τ^1, τ^2, τ^3 of a sought 11-cycle belong to the first copy, other four vertices $\pi^1, \pi^2, \pi^3, \pi^4$ belong to the second copy, and remaining four vertices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$ belong to the third copy. Let $\pi^1 = \tau^1 r_n = [n\,n-1\,\ldots\,2\,1]$, $\pi^4 = \gamma^4 r_n$, $\tau^3 = \gamma^4 r_n$, then γ^1 and γ^4 should belong to either $P_{n-1}(n-1)$ or $P_{n-1}(3)$, since either $\gamma^1 = \tau^1 r_{n-1} r_{n-3} r_n$ or $\gamma^1 = \tau^1 r_{n-3} r_{n-1} r_n$, where

$$\gamma^1 = \begin{cases} [n\,n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1] = \gamma^1(A) & \text{or} \\ [n\,1\,2\,n-1\,n-2\,\ldots\,4\,3] = \gamma^1(B). \end{cases} \tag{2.34}$$

On the other hand, it is obvious that there are two ways to get a path of length five from τ^1 to γ^5:

$$\gamma^4 = \begin{cases} \tau^1\,r_n\,r_{n-3}\,r_{n-1}\,r_{n-3}\,r_n = [1\,4\,5\,2\,3\,n\,n-1\ldots\,7\,6] = \gamma^4(C) & \textit{or} \\ \tau^1\,r_n\,r_{n-1}\,r_{n-3}\,r_{n-1}\,r_n = [1\,n-2\,\ldots\,3\,2\,n-1\,n] = \gamma^4(D). \end{cases} \tag{2.35}$$

To get a sought 11-cycle there should be a path of length four between γ^1 and γ^4. Let us check this. If $n = 5$ then by (2.34) and (2.35) we have $\gamma^1(B) = [5\,1\,2\,4\,3]$ and $\gamma^4(C) = [1\,4\,5\,2\,3]$, but there is no path of length three between them, since we have either

$$\gamma^4(C)\,r_2\,r_4\,r_2 = [5\,2\,1\,4\,3] \neq \gamma^1(B) \ \text{ or } \ \gamma^4(C)\,r_4\,r_2\,r_4 = [1\,4\,2\,5\,3] \neq \gamma^1(B).$$

Hence, a path of length three does not occur between γ^4 and γ^1.

If $n = 7$ then by (2.34) and (2.35) we have $\gamma^1(A) = [7\,4\,3\,2\,1\,5\,6]$ and $\gamma^4(C) = [1\,4\,5\,2\,3\,7\,6]$, but there is no path of length three between them, since we have either

$$\gamma^4(C)\, r_4\, r_6\, r_4 = [4\,1\,3\,7\,5\,2\,6] \neq \gamma^1(A) \quad \text{or} \quad \gamma^4(C)\, r_6\, r_4\, r_6 = [1\,4\,7\,3\,2\,5\,6] \neq \gamma^1(A).$$

Again, a path of length three does not occur between γ^4 and γ^1.

If $n \geqslant 9$ then there is a contradiction with an assumption that both γ^1 and γ^4 belong to either $P_{n-1}(n-1)$ or $P_{n-1}(3)$. Hence, a sought cycle cannot occur in this case.

Case 3: 11-cycle within P_n has vertices from four copies of P_{n-1}.

There are three possible situations in this case.

Case $(\mathbf{2+2+2+5})$. Suppose that two vertices π^1, π^2 of a sought 11-cycle belong to the first copy, two vertices τ^1, τ^2 belong to the second copy, two vertices γ^1, γ^2 belong to the third copy and remaining five vertices $\sigma^1, \sigma^2, \sigma^3, \sigma^4, \sigma^5$ belong to the fourth copy. Let $\tau^1 = \gamma^1 r_n$, $\gamma^2 = \sigma^1 r_n$, $\sigma^5 = \pi^2 r_n$ and $\pi^1 = \tau^2 r_n$, then σ^1 and σ^5 should belong to either $P_{n-1}(2)$ or $P_{n-1}(4)$, since either $\sigma^1 = \tau^1 r_n r_{n-1} r_n$ or $\sigma^1 = \tau^1 r_n r_{n-3} r_n$ where

$$\sigma^1 = \begin{cases} [1\,n\,n-1\,n-2\,\ldots\,3\,2] = \sigma^1(A) & \text{or} \\ [1\,2\,3\,n\,n-1\,\ldots\,5\,4] = \sigma^1(B). \end{cases} \tag{2.36}$$

Taking into account the same arguments as we used in the proof of Theorem 4, Case $(2+2+2+4)$, one can conclude that there are four ways to reach σ^5 by a path of length four from $\tau^1 = I_n$ such that we have:

$$\sigma^5 = \begin{cases} [n-3\,n-4\,n-5\,n\,n-1\,n-2\,1\,2\,\ldots\,n-7\,n-6] = \sigma^5(C) & \text{or} \\ [n-1\,n-2\,n-3\,n\,1\,2\,\ldots\,n-5\,n-4] = \sigma^5(D) & \text{or} \\ [n-3\,n\,n-1\,n-2\,1\,2\,\ldots\,n-5\,n-4] = \sigma^5(E) & \text{or} \\ [n-1\,n\,1\,2\,\ldots\,n-3\,n-2] = \sigma^5(F). \end{cases} \tag{2.37}$$

To get a sought 11-cycle there should be a path of length four between σ^1 and σ^5. Let us check this. If $n = 5$ then by (2.36) we have $\sigma^1(B) = [1\,2\,3\,5\,4]$, and $\sigma^5(C) = I_n(r_{n-3}\, r_n)^2 = [2\,1\,3\,5\,4]$, but there is no path of length four between them, since we have either

$$\sigma^5(C)(r_2\, r_4)^2 = [1\,2\,5\,3\,4] \neq \sigma^1(B) \quad \text{or} \quad \sigma^5(C)(r_4\, r_2)^2 = [2\,1\,5\,3\,4] \neq \sigma^1(B).$$

If $n \geqslant 7$ then by (2.36) and (2.37) there is a contradiction with an assumption that both σ^1 and σ^5 belong to either $P_{n-1}(2)$ or $P_{n-1}(4)$. Hence, a sought cycle cannot occur in this case.

Case $(\mathbf{2+3+3+3})$. Suppose that three vertices τ^1, τ^2, τ^3 of a sought 11-cycle belong to the first copy, three vertices π^1, π^2, π^3 belong to the second copy, three vertices $\gamma^1, \gamma^2, \gamma^3$ belong to the third copy and remaining two vertices σ^1, σ^2 belong to the fourth copy. Let $\tau^1 = \pi^1 r_n$, $\pi^3 = \sigma^2 r_n$, $\tau^3 = \gamma^1 r_n$ and $\gamma^3 = \sigma^1 r_n$. Taking into account the same arguments as we used in the proof of Theorem 4, Case $(3+3+4)$, one can conclude that there are two ways to reach σ^2 by a path of length three from $\tau^1 = I_n$ such that we have:

$$\sigma^2 = \begin{cases} \tau^1\, r_n\, r_{n-3}\, r_{n-1}\, r_{n-3}\, r_n = [1\,4\,5\,2\,3\,n\,n-1\ldots\,7\,6] = \sigma^2(A) & \text{or} \\ \tau^1\, r_n\, r_{n-1}\, r_{n-3}\, r_{n-1}\, r_n = [1\,n-2\,\ldots\,3\,2\,n-1\,n] = \sigma^2(B). \end{cases} \tag{2.38}$$

On the other hand, by (2.26) and (2.27), and since $\gamma_1 = \tau^3 r_n$, there are two ways to get paths of length three from τ^1 to γ^1 such that either

$$\gamma^1 = [n\,n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1] \quad \text{or} \quad \gamma^1 = [n\,1\,2\,n-1\,n-2\,\ldots\,4\,3].$$

Then there are two ways to get paths of length two from γ^1 to γ^3 such that the first way is presented as follows:

$$\gamma^1 r_{n-3} r_{n-1} = \gamma^3.$$

Namely, we get either

$$\gamma^1 = [n\,n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1] \xrightarrow{r_{n-3}} [2\,3\,\ldots\,n-4\,n-3\,n\,1\,n-2\,n-1] \xrightarrow{r_{n-1}}$$

$$[n-2\,1\,n\,n-3\,n-4\,\ldots\,3\,2\,n-1] = \gamma^3$$

or

$$\gamma^1 = [n\,1\,2\,n{-}1\,n{-}2\,\ldots\,4\,3] \xrightarrow{r_{n-3}} [6\,7\ldots n{-}2\,n{-}1\,2\,1\,n\,5\,4\,3] \xrightarrow{r_{n-1}} [4\,5\,n\,1\,2\,n{-}1\,n{-}2\,\ldots\,6\,5\,3] = \gamma^3.$$

The second way is presented as follows: $\gamma^1 r_{n-1} r_{n-3} = \gamma^3$. Namely, we get

$$\gamma^1 = [n\,n-3\,n-4\,\ldots\,2\,1\,n-2\,n-1] \xrightarrow{r_{n-1}} [n-2\,1\,2\,\ldots\,n-4\,n-3\,n\,n-1] \xrightarrow{r_{n-3}}$$

$$[n-4\,n-5\,\ldots\,2\,1\,n-2\,n-3\,n\,n-1] = \gamma^3$$

or

$$\gamma^1 = [n\,1\,2\,n-1\,n-2\,\ldots\,4\,3] \xrightarrow{r_{n-1}} [4\,5\,\ldots\,n-2\,n-1\,2\,1\,n\,3] \xrightarrow{r_{n-3}} [2\,n-1\,n-2\,\ldots\,5\,4\,1\,n\,3] = \gamma^3.$$

Since $\sigma^1 = \gamma^3 r_n$, we get

$$\sigma^1 = \begin{cases} [n-1\,2\,3\,\ldots\,n-4\,n-3\,n\,1\,n-2] = \sigma^1(C) & \text{or} \\ [3\,5\,6\,\ldots\,n-2\,n-1\,2\,1\,n\,5\,4] = \sigma^1(D) & \text{or} \\ [n-1\,n\,n-3\,n-2\,1\,2\,\ldots\,n-5\,n-4] = \sigma^1(E) & \text{or} \\ [3\,n\,1\,4\,5\,\ldots\,n-1\,2] = \sigma^1(F). \end{cases} \tag{2.39}$$

To get a sought 11-cycle vertices σ^1 and σ^2 should be adjacent by an internal edge. However, if $n = 5$ then by (2.39) and (2.38), we have two non-adjacent vertices $\sigma^1(C) = [4\,2\,5\,1\,3]$ and $\sigma^2(A) = [1\,4\,5\,2\,3]$. If $n \geqslant 7$ then by (2.38) and (2.39) there is a contradiction with an assumption that σ^1 and σ^2 belong to the same copy. Hence, a sought cycle cannot occur in this case.

Case $(\mathbf{2+2+3+4})$. There are two subcases due to a sequence of vertex from the copies forming a cycle: 1) (2,3,4,2); 2) (3,2,4,2).

Subcase 1. Suppose that four vertices $\pi^1, \pi^2, \pi^3, \pi^4$ of a sought 11-cycle belong to the first copy, two vertices τ^1, τ^2 belong to the second copy, three vertices $\gamma^1, \gamma^2, \gamma^3$ belong to the third copy and remaining two vertices σ^1, σ^2 belong to the fourth copy. Taking into account the same arguments as we used in the proof of Theorem 4, Case $(2+3+2+3)$, one can conclude that there are four ways to reach σ^4 by a path of length five from $\tau^1 = I_n$:

$$\sigma^4 = \begin{cases} [n-3\,n-6\,n-7\,\ldots\,3\,2\,1\,n-2\,n-1\,n\,n-5\,n-4] = \sigma^4(A) & \text{or} \\ [n-1\,n-4\,n-5\,\ldots\,3\,2\,1\,n\,n-3\,n-2] = \sigma^4(B) & \text{or} \\ [n-3\,n\,n-1\,n-4\,n-5\,\ldots\,3\,2\,1\,n-2] = \sigma^4(C) & \text{or} \\ [n-1\,n\,1\,n-2\,n-3\,\ldots\,3\,2] = \sigma^4(D). \end{cases}$$

On the other hand, there are two ways to reach σ^1 by a path of length three from τ^1 such that we have:

$$\sigma^1 = \begin{cases} [1\,2\,3\,n\,n-1\,n-2\,\ldots\,6\,5\,4] = \sigma^1(E) & \text{or} \\ [1\,n\,n-1\,n-2\,\ldots\,3\,2] = \sigma^1(F). \end{cases}$$

As one can see, vertices $\sigma^1(F)$ and $\sigma^4(D)$ belong to the same copy $P_{n-1}(2)$. Let us check whether there is a path of length three between these two vertices. Indeed, there are two ways to get a path of length three from $\sigma^1(F)$ to $\sigma^4(D)$. The first way is presented as follows:

$$\sigma^1 r_{n-1} r_{n-3} r_{n-1} = \sigma^4, \tag{2.40}$$

where

$$\sigma^1(F) = [1\,n\,n-1\,\ldots\,3\,2] \xrightarrow{r_{n-1}} [3\,4\,\ldots\,n-1\,n\,1\,2] \xrightarrow{r_{n-3}} [n-1\,n-2\,\ldots\,4\,3\,n\,1\,2] \xrightarrow{r_{n-1}}$$

$$[1\,n\,3\,4\,\ldots\,n-2\,n-1\,2] \neq \sigma^4(D).$$

The second way is presented as follows:

$$\sigma^1 r_{n-3} r_{n-1} r_{n-3} = \sigma^4, \tag{2.41}$$

where

$$\sigma^1(F) = [1\,n\,n-1\,\ldots\,3\,2] \xrightarrow{r_{n-3}} [5\,6\,\ldots\,n-1\,n\,1\,4\,3\,2] \xrightarrow{r_{n-1}} [3\,4\,1\,n\,n-1\,\ldots\,6\,5\,2] \xrightarrow{r_{n-3}}$$

$$[7\,8\,\ldots\,n-1\,n\,1\,4\,3\,6\,5\,2] \neq \sigma^4(D).$$

Thus, a sought cycle cannot occur in this subcase.

Subcase 2. Suppose that three vertices τ^1, τ^2, τ^3 of a sought 11-cycle belong to the first copy, two vertices π^1, π^2 belong to the second copy, four vertices $\sigma^1, \sigma^2, \sigma^3, \sigma^4$ belong to the third copy and remaining two vertices γ^1, γ^2 belong to the fourth copy. Let $\pi^1 = \tau^3 r_n$, $\pi^2 = \sigma^4 r_n$ and $\tau^1 = \gamma^1 r_n$, $\gamma^2 = \sigma^1 r_n$. Taking into account the same arguments as we used in the proof of Theorem 4, Case $(3+2+3+2)$, one can conclude that there are four ways to reach σ^4 by a path of length five from $\tau^1 = I_n$:

$$\sigma^4 = \begin{cases} [n-1\,n-2\,1\,n\,n-3\,n-4\,\ldots\,3\,2] = \sigma^4(A) & \text{or} \\ [3\,4\,5\,n\,1\,2\,n-1\,n-2\,\ldots\,7\,6] = \sigma^4(B) & \text{or} \\ [n-1\,n\,n-3\,n-4\,\ldots\,2\,1\,n-2] = \sigma^4(C) & \text{or} \\ [3\,n\,1\,2\,n-1\,n-2\,\ldots\,5\,4] = \sigma^4(D). & \end{cases}$$

On the other hand, there are two ways to reach σ^1 by a path of length three from τ^1 such that we have:

$$\sigma^1 = \begin{cases} [1\,2\,3\,n\,\ldots\,6\,5\,4] = \sigma^1(E) & \text{or} \\ [1\,n\,n-1\,n-2\,\ldots\,3\,2] = \sigma^1(F). & \end{cases}$$

It is easy to see that vertices $\sigma^1(E)$ and $\sigma^4(D)$ belong to the copy $P_{n-1}(4)$. However, there is no path of length three between these two vertices. Indeed, by (2.40) and (2.41), there are two ways to get a path of length three from $\sigma^1(E)$ to $\sigma^4(D)$ such that either

$$\sigma^1(E) = [1\,2\,3\,n\,\ldots\,6\,5\,4] \xrightarrow{r_{n-1}} [5\,6\,\ldots\,n-1\,n\,3\,2\,1\,4] \xrightarrow{r_{n-3}} [3\,n\,n-1\,\ldots\,6\,5\,2\,1\,4] \xrightarrow{r_{n-1}}$$

$$[1\,2\,5\,6\,\ldots\,n-1\,n\,3\,4] \neq \sigma^4(D), \quad \text{or}$$

or

$$\sigma^1(E) = [1\,2\,3\,n\,\ldots\,6\,5\,4] \xrightarrow{r_{n-3}} [7\,8\,\ldots\,n-1\,n\,3\,2\,1\,6\,5\,4] \xrightarrow{r_{n-1}} [5\,6\,1\,2\,3\,n\,n-1\,\ldots\,8\,7\,4] \xrightarrow{r_{n-3}}$$

$$[9\,10\,\ldots\,n-1\,n\,3\,2\,1\,6\,5\,8\,7\,4] \neq \sigma^4(D).$$

Hence, there is no path of length three between these two vertices.

One can also see that vertices $\sigma^1(F)$ and $\sigma^4(A)$ belong to the copy $P_{n-1}(2)$. Let us check whether there is a path of length three between these two vertices. By (2.40) and (2.41), there are two ways to get a path of length three from $\sigma^1(F)$ to $\sigma^4(A)$ such as either

$$\sigma^1(F) = [1\,n\,n-1\,\ldots\,3\,2] \xrightarrow{r_{n-1}} [3\,4\,\ldots\,n-1\,n\,1\,2] \xrightarrow{r_{n-3}} [n-1\,n-2\,\ldots\,4\,3\,n\,1\,2] \xrightarrow{r_{n-1}}$$

$$[1\,n\,3\,4\,\ldots\,n-2\,n-1\,2] \neq \sigma^4(A),$$

or

$$\sigma^1(F) = [1\,n\,n-1\,\ldots\,3\,2] \xrightarrow{r_{n-3}} [5\,6\,\ldots\,n-1\,n\,1\,4\,3\,2] \xrightarrow{r_{n-1}} [3\,4\,1\,n\,n-1\,\ldots\,6\,5\,2] \xrightarrow{r_{n-3}}$$

$$[7\,8\,\ldots\,n-1\,n\,1\,4\,3\,6\,5\,2] \neq \sigma^4(A).$$

Thus, a sought cycle cannot occur in this subcase.

Case 4: 11-cycle within P_n has vertices from five copies of P_{n-1}.

There is the only possible situation in this case.

Case $(\mathbf{2+2+2+2+3})$. Suppose that two vertices π^1, π^2 of a sought 11-cycle belong to the first copy, two vertices τ^1, τ^2 belong to the second copy, two vertices γ^1, γ^2 belong to the third copy, three vertices $\delta^1, \delta^2, \delta^3$ belong to the fourth copy and remaining two vertices σ^1, σ^2 belong to the fifth copy. Let $\pi^1 = \tau^2 r_n$, $\pi^2 = \delta^3 r_n$, $\delta^1 = \sigma^2 r_n$, $\sigma^1 = \gamma^2 r_n$ and $\gamma^1 = \tau^1 r_n$. Taking into account the same arguments as we used in the proof of Theorem 4, Case $(2+2+2+2+2)$, one can conclude that there are four ways to reach δ^3 by a path of length four from $\tau^1 = I_n$:

$$\delta^3 = \begin{cases} [n-3\,n-4\,n-5\,n\,n-1\,n-2\,1\,2\,\ldots\,n-7\,n-6] & \text{or} \\ [n-1\,n-2\,n-3\,n\,1\,2\,\ldots\,n-5\,n-4] & \text{or} \\ [n-3\,n\,n-1\,n-2\,1\,2\,\ldots\,n-5\,n-4] & \text{or} \\ [n-1\,n\,1\,2\,\ldots\,n-3\,n-2]. \end{cases} \tag{2.42}$$

On the other hand, there are four ways to reach δ^1 by a path of length five from τ^1 such that we have:

$$\delta^1 = \begin{cases} [4\,5\,6\,1\,2\,3\,n\,n-1\,\ldots\,8\,7] & \text{or} \\ [2\,3\,4\,1\,n\,n-1\,\ldots\,6\,5] & \text{or} \\ [4\,1\,2\,3\,n\,n-1\,\ldots\,6\,5] & \text{or} \\ [2\,1\,n\,n-1\,\ldots\,4\,3]. \end{cases} \tag{2.43}$$

To get a sought 11-cycle, either $\delta^1 = \delta^3\, r_{n-3}\, r_{n-1}$ or $\delta^1 = \delta^3\, r_{n-1}\, r_{n-3}$ should hold. Let us check this. If $n=5$, then by (2.43) and (2.42) we have:

$$\delta^1 = \begin{cases} [4\,5\,3\,1\,2] = I_n (r_n\, r_{n-3})^2\, r_n & \text{or} \\ [2\,3\,4\,1\,5] & \text{or} \\ [4\,1\,2\,3\,5] & \text{or} \\ [2\,1\,5\,4\,3], \end{cases} \qquad \delta^3 = \begin{cases} [4\,3\,2\,5\,1] & \text{or} \\ [4\,5\,1\,2\,3] & \text{or} \\ [2\,1\,3\,5\,4] & \text{or} \\ [2\,5\,4\,3\,1]. \end{cases}$$

As one can see, there are no path of length two between δ^1 and δ^3. Thus, a sought 11-cycle can not occur in this case. If $n=7$, then by (2.43) and (2.42) we have:

$$\delta^1 = \begin{cases} [4\,5\,6\,1\,2\,3\,7] & \text{or} \\ [2\,3\,4\,1\,7\,6\,5] & \text{or} \\ [4\,1\,2\,3\,7\,6\,5] & \text{or} \\ [2\,1\,7\,6\,5\,4\,3], \end{cases} \qquad \delta^3 = \begin{cases} [4\,3\,2\,7\,6\,5\,1] & \text{or} \\ [6\,5\,4\,7\,1\,2\,3] & \text{or} \\ [4\,7\,6\,5\,1\,2\,3] & \text{or} \\ [6\,7\,1\,2\,3\,4\,5]. \end{cases}$$

Again, a sought 11-cycle can not occur in this case, since there is no path of length two between δ^1 and δ^3. By using similar arguments, one can check that for $n = 9, 11, 13$ there is no 11-cycle in the graph. For any odd $n \geqslant 15$, there is a contradiction with an assumption that both δ^1 and δ^3 belong to the same copy. This complete the proof of the last case of Theorem 5. $\square$

3. Proof of Theorem 1

It is obvious that any cycle of the cubic pancake graphs P_n^i, $i = 1, \dots, 5$, does belong to the pancake graph $P_n, n \geqslant 4$, and should be described by one of the canonical formulas from Theorem 2 and Theorem 3. Let us check which cycles appear in P_n^i for each $i \in \{1, \dots, 5\}$.

Case (1.1). Any cycle of $P_n^1, n \geqslant 4$, is formed by prefix-reversals from the set $BS_1 = \{r_2, r_{n-1}, r_n\}$. If $n = 4$ then by Theorem 2 the prefix-reversals r_2 and r_3 give 6-cycles of the form (2.1). For $n \geqslant 4$, by (2.2) there are no 7-cycles in P_n^1, but there are 8-cycles of the form (2.4) when $n = 4$ and there are 8-cycles of the form (2.9) for $n \geqslant 5$ if we put $k = n$, $i = 2$, $j = n - 1$. The canonical form in the last case is given by $(r_n\, r_2)^4$ for any $n \geqslant 5$. Hence, the formula (1.1) holds.

Using similar arguments, one can see that any cycle of $P_n^2, n \geqslant 4$, is presented by prefix-reversals from the set $BS_2 = \{r_{n-2}, r_{n-1}, r_n\}$. If $n = 4$ then obviously P_4^2 has 6-cycles. If $n > 4$ then by (2.2) there are no 7-cycles in P_n^2, however by Theorem 3 there are 8-cycles of the canonical form (2.3). Indeed, if we put $k = n$, $j = n - 1$, $i = n - 2$ in (2.3) then we have the sequence $r_n\, r_{n-1}\, r_{n-2}\, r_{n-1}\, r_n\, r_{n-1}\, r_{n-2}\, r_{n-1}$. Thus, the formula (1.1) holds in this case.

The same arguments appear for the graph $P_n^3, n \geqslant 4$ is even, whose generating set contains prefix-reversals r_3, r_{n-2} and r_n. It has 6-cycles of the form (2.1) and 8-cycles of the form (2.10) if $n = 4$, but it does not have 7-cycles for $n \geqslant 4$. Moreover, for any even $n \geqslant 6$ in P_n^3 there are no 6-cycles, but there are 8-cycles of the form (2.9) if we put $k = n$, $i = 3$, $j = n - 2$. The canonical form of 8-cycles in this case is given by $(r_n\, r_3)^4$ for any even $n \geqslant 6$. Thus, the formula (1.1) holds.

Case (1.2). In the case of the graph $P_n^4, n \geqslant 5$ is odd, its generating elements r_3, r_{n-1} and r_n give 8-cycles of the canonical form $(r_n\, r_{n-1}\, r_n\, r_3)^2$ if we put $k = n$, $j = 2$ and $i = 3$ in the form (2.9). Obviously, there are no 6-cycles in the graph since r_2 does not belong to the generating set for any $n \geqslant 5$. Hence, the formula (1.2) holds for any odd $n \geqslant 5$.

Case (1.3). The generating set $BS_5 = \{r_{n-3}, r_{n-1}, r_n\}$ of the graph P_n^5, where $n \geqslant 5$ is odd, gives the canonical form $(r_5\, r_4\, r_5\, r_2)^2$ of 8-cycles if we put $k = 5$, $j = 2$ and $i = 2$ in the form (2.9). By Theorem 2 there are no 6-cycles in the graph for any $n \geqslant 5$. By Theorem 3 and by the characterization of 9-cycles in the pancake graph [6, Theorem 4] there are no 8- and 9-cycles in P_n^5 for any $n \geqslant 7$. By Theorem 4 and Theorem 5 for any $n \geqslant 7$ there are no 10-cycles and 11-cycles in P_n^5, and the smallest cycle in P_n^5 is 12-cycle of the canonical form $C_{12} = (r_n\, r_{n-1}\, r_n\, r_{n-1}\, r_{n-3}\, r_{n-1})^2$.

This complete the proof of Theorem 1. $\square$

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