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FINITE SOLVABLE GROUPS WHOSE GRUENBERG–KEGEL GRAPHS ARE ISOMORPHIC TO THE PAW [1,2]

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The Gruenberg–Kegel graph (or the prime graph) of a finite group G is the graph, in which the vertex set is the set of all prime divisors of the order of G and two different vertices p and q are adjacent if and only if there exists an element of order pq in G. The paw is the graph on four vertices whose degrees are 1, 2, 2, and 3. We consider the problem of describing finite groups whose Gruenberg–Kegel graphs are isomorphic as abstract graphs to the paw. For example, the Gruenberg–Kegel graphs of the groups A_{10} and $\mathrm{Aut}(J_2)$ are isomorphic as abstract graphs to the paw. In this paper, we describe finite solvable groups whose Gruenberg–Kegel graphs are isomorphic as abstract graphs to the paw.

Keywords: finite group, solvable group, Gruenberg–Kegel graph, paw.

А. С. Кондратьев, Н. А. Минигулов. Конечные разрешимые группы, графы Грюнберга — Кегеля которых изоморфны графу “балалайка”.

Граф Грюнберга — Кегеля (или граф простых чисел) конечной группы G — это граф, в котором вершинами служат все простые делители порядка группы G и две различные вершины p и q смежны тогда и только тогда, когда G содержит элемент порядка pq. Граф “балалайка” — это граф на четырех вершинах, степени которых равны 1, 2, 2 и 3. Мы рассматриваем проблему описания конечных групп, графы Грюнберга — Кегеля которых как абстрактные графы изоморфны графу “балалайка”. Например, графы Грюнберга — Кегеля групп A_{10} и $\mathrm{Aut}(J_2)$ как абстрактные графы изоморфны графу “балалайка”. В этой работе мы описываем конечные разрешимые группы, графы Грюнберга — Кегеля которых изоморфны графу “балалайка”.

Ключевые слова: конечная группа, разрешимая группа, граф Грюнберга — Кегеля, граф "балалайка".

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Introduction

The *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of a finite group G is the graph, in which the vertex set is the set of all prime divisors of the order of G and two different vertices p and q are adjacent if and only if there exists an element of order pq in G. The paw is the graph on four vertices whose degrees are 1, 2, 2, and 3.

The first author has described finite groups that have the same Gruenberg–Kegel graphs as the groups $\mathrm{Aut}(J_2)$ (see [4]) and A_{10} (see [5]). The Gruenberg–Kegel graphs of all these groups are isomorphic as abstract graphs to the paw.

We pose the following more general problem: describe finite groups whose Gruenberg–Kegel graphs are isomorphic as abstract graphs to the paw.

As a part of the solution of this problem, we proved in [6] that if G is a finite non-solvable group and the graph $\Gamma(G)$ as an abstract graph is isomorphic to the paw, then the quotient group $G/S(G)$ (where $S(G)$ is the solvable radical of G) is almost simple; we also classified all finite almost simple groups whose Gruenberg–Kegel graphs as abstract graphs are isomorphic to subgraphs of the paw.

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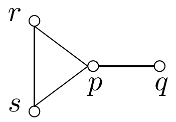
[2]This paper is based on the results of the 2021 Conference of International Mathematical Centers “Groups and Graphs, Semigroups and Synchronization”.

We also classified in [7] finite non-solvable groups G whose Gruenberg–Kegel graphs are isomorphic as abstract graphs to the paw in the followings two cases:

(1) G has no elements of order 6;

(2) G contains an element of order 6 and the vertex of degree 1 in the graph $\Gamma(G)$ divides $|S(G)|$.

In this paper, we describe finite solvable groups whose Gruenberg–Kegel graphs are isomorphic as abstract graphs to the paw. Let G be a finite solvable group whose Gruenberg–Kegel graph as an abstract graph is isomorphic to the paw; i.e., $\Gamma(G)$ has the form



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where r, s, p, and q are some pairwise distinct primes. By the P. Hall's theorem [1, Theorem 18.5], G has a p-complement L; i.e., L is a Hall $\{r,s,q\}$-subgroup of G. It is easy to see that the graph $\Gamma(L)$ is disconnected and has the form $r - s \quad q$. By the Gruenberg–Kegel theorem (see Lemma 1 below), L is either a Frobenius group or a 2-Frobenius group. Set $\pi = \{p,q\}$, $\overline{G} = G/O_\pi(G)$, and $\widetilde{G} = G/O_{q'}(G)$. We prove the following two theorems.

Theorem 1. *Let L be a 2-Frobenius group of the form $L = A \rtimes (B \rtimes C)$, where $A = F(L)$, and $B = F(BC)$. Then one of the following statements holds*:

(1) *$O_p(G) < O_\pi(G)$, A is a Sylow q-subgroup of $O_\pi(G)$, B is a cyclic Hall $\{r,s\}$-subgroup of G, r and s are odd, C is a cyclic q-subgroup, and $\overline{G}$ is isomorphic to a subgroup of* $\mathrm{Hol}(\overline{B})$; *in particular, $\overline{G}/\overline{B}$ is an abelian $\{p,q\}$-group*;

(2) *$O_\pi(G) = O_p(G)$, A is a nilpotent Hall $\{r,s\}$-subgroup of $O_{q'}(G)$, B is a cyclic Sylow q-subgroup of G, C is a cyclic $\{r,s\}$-subgroup, $F(\widetilde{G}) = \widetilde{B}$, and $\widetilde{G}$ is isomorphic to a subgroup of* $\mathrm{Hol}(\widetilde{B})$; *in particular, $\widetilde{G}/\widetilde{B}$ is an abelian q'-group.*

Theorem 2. *Let L be a Frobenius group of the form $L = A \rtimes B$, where $A = F(L)$. Then one of the following statements holds*:

(1) *$O_p(G) < O_\pi(G)$, A is a Sylow q-subgroup of $O_\pi(G)$, B is a Hall $\{r,s\}$-subgroup of G, and one of the following statements holds*:

(1*a*) *r and s are odd, the Sylow subgroups of B are cyclic, B is metacyclic, $F(\overline{G}) = F(\overline{B})$ is a cyclic subgroup of order divisible by rs, and $\overline{G}$ is isomorphic to a subgroup of* $\mathrm{Hol}(\overline{B})$; *in particular, $\overline{G}/\overline{B}$ is an abelian group*;

(1*b*) *it can be assumed that $r = 2$, A is abelian, a Sylow 2-subgroup B_2 of B is a cyclic or (generalized) quaternion group, $O(B)$ is the cyclic Sylow s-subgroup of B, and one of the following statements holds*:

(1*bi*) *$G = O(G)B_2$, $O_\pi(G) = O_\pi(O(G))$, $F(\overline{O(G)}) = \overline{O(B)}$, and $O(\overline{G})$ is isomorphic to a subgroup of* $\mathrm{Hol}(\overline{O(B)})$; *in particular, $\overline{O(G)}/\overline{O(B)}$ is a cyclic p-group*;

(1*bii*) *$G/O(G)$ is isomorphic to $SL_2(3)$ or $SL_2(3){\cdot}2$, $B = O(B)B_2$, $p = 3$, $s > 3$, and statement* (1*bi*) *holds for the group $O(G)B_2$*;

(1*biii*) *$G/O(G) \cong B/O(B)$ is isomorphic to $SL_2(3)$ or $SL_2(3){\cdot}2$, $s = 3$, $p > 3$, $G = O_\pi(G)B$, $O(G) = O_\pi(G)O(B)$, and $O_\pi(G) = O_{q',q,q'}(O_\pi(G))$*;

(2) *$O_\pi(G) = O_p(G)$, A is a nilpotent $\{r,s\}$-Hall subgroup of G, B is a Sylow q-subgroup of G, and one of the following statements holds*:

(2*a*) *B is a cyclic group, $A \le O_{q'}(G)$, $F(\widetilde{G}) = \widetilde{B}$, and $\widetilde{G}$ is isomorphic to a subgroup of* $\mathrm{Hol}(\widetilde{B})$; *in particular, $\widetilde{G}/\widetilde{B}$ is an abelian q'-group*;

(2*b*) *B is a (generalized) quaternion group,* $q = 2$*, A is an abelian Hall* $\{r, s\}$*-subgroup of* $O(G)$*,* $F(\overline{G}) = \overline{A}$ *and either* $G = O(G)B$*, or* $p = 3$ *and* $G/O(G)$ *is isomorphic to* $SL_2(3)$ *or* $SL_2(3){\cdot}2$*.*

Remark. It is clear that $\Gamma(G) = \Gamma(\mathbb{Z}_p \times L)$ in Theorems 1 and 2. Using well-known properties of finite 3-primary solvable Frobenius groups and a criterion of the existence of finite 3-primary 2-Frobenius groups from [9, Proposition 1], we can show that all statements of Theorems 1 and 2 are realizable for some primes r, s, p and q.

1. Preliminaries

Our notation and terminology are mostly standard and can be found in [1–3]. A finite group G is called a *Frobenius* group with kernel A and complement B if $G = A \rtimes B$, where the groups A and B are non-trivial and $C_A(b) = 1$ for any non-trivial element b of B. A finite group G is called a 2-*Frobenius* group if there exist subgroups A, B, and C of G such that $G = ABC$, A and AB are normal subgroups of G, and AB and BC are Frobenius groups with kernels A and B and complements B and C, respectively. If G is a group, then the natural semi-direct product $G \rtimes \mathrm{Aut}(G)$ is called the *holomorph* of G and is denoted by $\mathrm{Hol}(G)$.

Let us recall some results, which are used in the proofs of the theorems.

Lemma 1 (the Gruenberg–Kegel theorem, see [8, Theorem A]). *If G is a finite group with disconnected Gruenberg–Kegel graph, then one of the following statements holds*:

(1) *G is a Frobenius group*;

(2) *G is a* 2*-Frobenius group*;

(3) *G is an extension of a nilpotent group by a group A, where* $\mathrm{Inn}(P) \le A \le \mathrm{Aut}(P)$ *for a simple non-abelian group P with disconnected Gruenberg–Kegel graph.*

Lemma 2 (see [3, Remark on p. 377]). *Let G be a finite group whose Sylow* 2*-subgroups are isomorphic to a (generalized) quaternion group, and let* $\overline{G} = G/O(G)$*. Then one of the following statements holds*:

(a) $\overline{G}$ *is isomorphic to a Sylow* 2*-subgroup of G*;

(b) $\overline{G}$ *is isomorphic to the group* $2{\cdot}A_7$;

(c) $\overline{G}$ *is an extension of the group* $SL_2(q)$*, where q is odd, by a cyclic group whose order is divisible by* 4*.*

2. Proof of Theorem 1

Let $L = A \rtimes (B \rtimes C)$ be a 2-Frobenius group, where $A = F(L)$ and $B = F(BC)$. By [9, Lemma 2], the subgroups B and C are cyclic. It is clear that $|B|$ is odd.

Suppose that q divides $|O_\pi(G)|$, and $Q \in Syl_q(O_\pi(G))$. Then $O_p(G) < O_\pi(G)$. By the Hall's theorem, we can assume that $Q \le L$. Hence $1 \ne Q = L \cap O_\pi(G) \trianglelefteq L$; in particular, $Q \le A = O_q(L)$.

If $q \notin \pi(C)$, then $\pi(BC) = \{r, s\}$. But the graph $\Gamma(BC)$ has the form $\overset{\circ\!-\!\circ}{r\ \ s}$; hence BC cannot be a Frobenius group, a contradiction.

Therefore, $q \in \pi(C)$; hence $\pi(C) = \{q\}$. It follows that $\pi(B) = \{r, s\}$. The subgroup B is a Hall $\{r, s\}$-subgroup of L, and hence of G.

We have $O_\pi(\overline{G}) = 1$; hence $1 \ne F(\overline{G}) \le O_{\pi'}(\overline{G}) = O_{\{r,s\}}(\overline{G})$. But $C_{\overline{G}}(F(\overline{G})) \le F(\overline{G})$; hence $F(\overline{G})$ is a cyclic $\{r, s\}$-Hall subgroup of $\overline{G}$; i. e., $F(\overline{G}) = \overline{B}$. It follows that $Q = A$, and statement (1) holds.

Suppose now that q does not divide $|O_\pi(G)|$. Then $O_\pi(G) = O_p(G)$. Since $O_\pi(\overline{G}) = 1$, we have $1 \neq F(\overline{G}) \leq O_{\pi'}(\overline{G}) = O_{\{r,s\}}(\overline{G})$. Clearly, $\overline{L} \cong L$; in particular, $\overline{L}$ is a p-complement in $\overline{G}$. Since $F(\overline{G}) \trianglelefteq \overline{L}$, we have $F(\overline{G}) \leq F(\overline{L})$; hence $\pi(F(\overline{L})) \subseteq \{r, s\}$.

If $q \notin \pi(B)$, then $\pi(AB) = \{r, s\}$. But the graph $\Gamma(AB)$ has the form $\underset{r}{\circ}\!\!-\!\!\underset{s}{\circ}$; hence AB cannot be a Frobenius group, a contradiction.

Therefore, $q \in \pi(B)$. Hence, B is a cyclic Sylow q-subgroup of G, $q > 2$, and $\pi(AC) = \{r, s\}$. We have $O_p(G) \leq O_{q'}(G)$, and $1 \neq F(\overline{G}) \leq O_{\{r,s\}}(\overline{G}) \leq \overline{O_{q'}(G)}$.

Let $B_0 \in Syl_q(O_{q',q}(G))$. Then we can assume that B_0 is a non-trivial subgroup of B. By [3, Theorem 6.3.3], $C_G(B_0) \leq O_{q',q}(G)$. Since B is a cyclic q-subgroup of G and $B \in C_G(B_0)$, $B = B_0$. Then $C_{\widetilde{G}}(\widetilde{B}) = \widetilde{B}$, and hence $\widetilde{B} = F(\widetilde{G})$. Therefore, A is a $\{r, s\}$-Hall subgroup of $O_{q'}(G)$, $\widetilde{G}$ is isomorphic to a subgroup of $\mathrm{Hol}(\widetilde{B})$, and $\widetilde{G}/\widetilde{B}$ is an abelian q'-group. Therefore, statement (2) holds.

Theorem 1 is proved.

3. Proof of Theorem 2

Let $L = A \rtimes B$ be a Frobenius group, where $A = F(L)$.

Suppose that q divides $|O_\pi(G)|$, and $Q \in Syl_q(O_\pi(G))$. Then $Q \neq 1$ and $O_p(G) < O_\pi(G)$. By the Hall's theorem, we can assume that $Q \leq L$. Hence, $1 \neq Q = L \cap O_\pi(G) \trianglelefteq L$; in particular, $Q \leq A = O_q(L)$. Therefore, B is a Hall $\{r, s\}$-subgroup of G. By [3, Theorem 10.3.1], the Sylow subgroups of B are either cyclic or (generalized) quaternion groups. We have $O_\pi(\overline{G}) = 1$, and $1 \neq F(\overline{G}) \leq O_{\pi'}(\overline{G}) = O_{\{r,s\}}(\overline{G})$. Hence $F(\overline{G}) \leq O_{\{r,s\}}(\overline{G}) \leq \overline{B}$. If $Q < A$, then $\overline{A} \neq 1$, and $[F(\overline{G}), \overline{A}] = 1$, a contradiction. Therefore, $Q = A$. If $2 \in \{r, s\}$, then, by [3, Theorem 10.3.1], the subgroup A is abelian.

Suppose that r and s are odd. Then Sylow subgroups of B are cyclic and, consequently, $F(\overline{G})$ is a normal cyclic $\{r, s\}$-subgroup of $\overline{G}$. By [3, Theorem 10.3.1], B is metacyclic and rs divides $|F(\overline{G})|$. Since $C_{\overline{G}}(F(\overline{G})) = F(\overline{G})$ and $\mathrm{Aut}(F(\overline{G}))$ is abelian, it follows that $\overline{G}/F(\overline{G})$ is abelian and $\overline{G}$ is isomorphic to a subgroup of $\mathrm{Hol}(F(\overline{G}))$. Therefore, statement ($1a$) holds.

Now we can assume that $r = 2$. A Sylow 2-subgroup B_2 of B is either a cyclic or (generalized) quaternion group. By the Burnside's theorem [3, Theorem 7.4.3] and Lemma 2, either $G = O(G)B_2$, or $G/O(G) \cong SL_2(3)$, or $G/O(G) \cong SL_2(3){\cdot}2$.

Suppose that $G = O(G)B_2$. Then $B = \langle x \rangle \rtimes B_2$, where $\langle x \rangle = B \cap O(G)$ is a Sylow s-subgroup of $O(G)$, and $C_{B_2}(x) \neq 1$. It is clear that $O_\pi(O(G)) = O_\pi(G)$. Arguing as above, we get $F(\overline{O(G)}) = \langle \overline{x} \rangle$, $\overline{O(G)}/\langle \overline{x} \rangle$ is an abelian group, and a group $\overline{O(G)}$ is isomorphic to subgroup of $\mathrm{Hol}(\langle \overline{x} \rangle)$, in particular, $\overline{O(G)}/\overline{O(B)}$ is a cyclic p-group. Therefore, statement ($1bi$) holds.

Now, we can assume that $G/O(G) \cong SL_2(3)$ or $SL_2(3){\cdot}2$.

Suppose that $B = O(B)B_2$. Then $p = 3$, $s > 3$, and statement ($1bi$) holds for the group $O(G)B_2$. Therefore, statement ($1bii$) holds.

Suppose that $B \neq O(B)B_2$. Then $G/O(G) \cong B/O(B)$. Hence, $s = 3$, $p, q > 3$, $B = B_2\langle x \rangle$, where $\langle x \rangle$ is a Sylow 3-subgroup of B, $O_2(B) \cong Q_8$, $F(B) = O_2(B) \times \langle x^3 \rangle$, and $\langle x^3 \rangle$ is a Sylow 3-subgroup of $O(G)$.

If $x^3 = 1$, then $\pi(O(G)) = \{p, q\}$; hence $O(G) = O_\pi(G)$ and $G = O_\pi(G) \rtimes B$.

Let $x^3 \neq 1$. Arguing as above, we get $O_\pi(G) = O_\pi(O(G))$, $F(\overline{O(G)}) = \langle \overline{x} \rangle$, and $\overline{O(G)}/\langle \overline{x} \rangle$ is isomorphic to a subgroup of $\mathrm{Aut}(\langle \overline{x} \rangle)$. But $\mathrm{Aut}(\langle \overline{x} \rangle)$ is a 2-group; hence $\overline{O(G)} = \langle \overline{x} \rangle$. Therefore, $G = O_\pi(G) \rtimes B$. Since A is an abelian Sylow q-subgroup of $O_\pi(G)$, we have, by [3, Theorem 6.3.3], $O_{q',q}(O_\pi(G)) = O_p(G)A$, and hence $O_\pi(G) = O_p(G)N_{O_\pi(G)}(Q)$ and $O_\pi(G) = O_{q',q,q'}(O_\pi(G))$. Therefore, statement ($1biii$) holds.

Suppose now that q does not divide $|O_\pi(G)|$. Then $O_\pi(G) = O_p(G)$. Since $O_\pi(\overline{G}) = 1$, we have $1 \neq F(\overline{G}) \leq O_{\pi'}(\overline{G}) = O_{\{r,s\}}(\overline{G})$. It is clear that $\overline{L} \cong L$; in particular, $\overline{L}$ is a p-complement in $\overline{G}$. Since $F(\overline{G}) \trianglelefteq \overline{L}$, we have $F(\overline{G}) \leq F(\overline{L})$; hence $\pi(F(L)) \subseteq \{r, s\}$. Therefore, $\pi(A) \subseteq \{r, s\}$, and $q \in \pi(B)$. Since B is a complement of a Frobenius group, we have $\pi(B) = \{q\}$, and hence $\pi(A) = \{r, s\}$. So, A is a nilpotent Hall $\{r, s\}$-subgroup of G, and B is a Sylow q-subgroup of G. By [3, Theorem 10.3.1], B is either a cyclic group or a (generalized) quaternion group.

Let B be a cyclic group. Then, arguing as above, we conclude that A is a Hall $\{r, s\}$-subgroup of $O_{q'}(G)$, the group $\widetilde{G} = G/O_{q'}(G)$ is isomorphic to a subgroup of $\mathrm{Hol}(\widetilde{B})$, and $\widetilde{G}/\widetilde{B}$ is an abelian q'-group. Therefore, statement $(2a)$ holds.

Let B be a (generalized) quaternion group. Then $q = 2$, and, arguing as above, we find that A is an abelian Hall $\{r, s\}$-subgroup of $O(G)$, $F(\overline{G}) = \overline{A}$, and either $G = O(G)B$ or $p = 3$ and $G/O(G)$ is isomorphic to $SL_2(3)$ or $SL_2(3){\cdot}2$. Therefore, statement $(2b)$ holds.

Theorem 2 is proved.

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