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ON SOME CONJECTURES RELATED TO QUANTITATIVE CHARACTERIZATIONS OF FINITE NONABELIAN SIMPLE GROUPS[1]

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In this note we provide some counterexamples for the conjecture of Moretó on finite simple groups, which says that any finite simple group G can be determined in terms of its order $|G|$ and the number of elements of order p, where p the largest prime divisor of $|G|$. A new characterization of all sporadic simple groups and alternating groups is given. Some related conjectures are also discussed.

Keywords: Finite simple groups, quantitative characterization, the largest prime divisor.

Цзиньбао Ли, Вуджи Ши. О некоторых гипотезах, связанных с числовой характеризацией конечных неабелевых простых групп.

В заметке приведены контрпримеры к гипотезе Морето, которая утверждает, что любая простая группа G может быть охарактеризована своим порядком и количеством элементов порядка p, где p — наибольший простой делитель порядка группы. Предложена новая характеризация всех спорадических простых групп и знакопеременных групп. Обсуждаются некоторые связанные гипотезы.

Ключевые слова: конечные простые группы, числовая характеризация, наибольший простой делитель.

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1. Introduction

All groups considered in this paper are finite and simple groups always mean finite nonabelian simple groups.

For a finite group G, let $\pi(G)$ be the set of all prime divisors dividing the order of G and let $|G|$ and $\pi_e(G)$ denote the order of G and the set of all element orders of G, respectively. For a positive number k, denote by $G(k)$ the set of all elements of order k in G. It is well known that $|G|$ and $\pi_e(G)$ are two of the most important quantitative invariants for G. In 1987, the second author proposed the following conjecture (see [12]).

Conjecture 1.1. *Let G be a group and S be a finite simple group. Then $G \cong S$ if and only if $\pi_e(G) = \pi_e(S)$ and $|G| = |S|$.*

J. G. Thompson said that *this would certainly be a nice theorem* (see [17], personal communication, January 4, 1988) if Shi's conjecture is true. From 1987 to 2003, the authors of [4; 12–16; 20] proved that this conjecture is correct for all finite simple groups except B_n, C_n and D_n (n even). At the end of 2009, the authors of [18] proved Shi's conjecture for the remaining difficult cases. Thus, this conjecture has been proved and becomes a theorem, that is, all finite simple groups can be determined by their orders and the sets of their element orders (briefly, 'two orders').

Recently, A. Moretó in [10] investigated the influence of the number of elements of order p in a given group G with $p \in \pi(G)$, and characterized some simple groups from this perspective. He showed that A_p and $L_2(p)$ are basically determined just by the number of elements of order p, where p is the largest prime divisor of the order of the group satisfying some additional conditions. Furthermore, Moretó posed the following conjecture.

[1]This paper is based on the results of the 2020 Ural Workshop on Group Theory and Combinatorics.

Conjecture 1.2. *Let S be a finite simple group and p the largest prime divisor of $|S|$. If G is a finite group with the same number of elements of order p as S and $|G| = |S|$, then $G \cong S$.*

In this note we first provide three counterexamples for the above conjecture.

Example 1.3. *Let $S = A_8 \cong L_4(2)$ and $G = L_3(4)$. Then we have $|S| = |G|$ and 7 is the largest prime in $\pi(S) = \pi(G)$. By [6], we have that both S and G contain $2^7 \cdot 3^2 \cdot 5$ elements of order 7. But $L_3(4)$ is not isomorphic to A_8.*

Example 1.4. *Let $S = O_7(3)$ and $G = S_6(3)$. We have $|S| = |G| = 2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$ and $p = 13$ is the largest prime dividing both the orders of S and G. Similarly, we may conclude from [6] that G and S have the same number of elements of order 13.*

Example 1.5. *Let $S = S_4(3) \cong U_4(2)$. Let A be a group of order 4, B be an elementary abelian group of order 2^4 and C be an elementary abelian group of order 3^4. Let D be a cyclic group of order 5 such that D acts trivially on A and acts fixed-point-freely on B and C, and set $G = (A \times B \times C) \rtimes D$. Then we have $|S| = |G|$ and $|S(5)| = |G(5)|$, which shows that S is not uniquely determined by its order and number of elements of order 5.*

Although Moretó's conjecture is not true in general, we can show that this conjecture holds for several classes of simple groups. For example, invoking Theorem 3.2 in [8] by A. Khosravi, B. Khosravi, it is easy to deduce that Conjecture 1.2 holds for sporadic simple groups. In fact, if G is a group and S is a sporadic simple groups satisfy the hypothesis of Conjecture 1.2, then by [6], one can compute the number of elements of order p, denoted by $|G(p)|$, where p is the largest prime in $\pi(G) = \pi(S)$. Let P be a Sylow p-subgroup of G. Then P is of order p and

$$|G : N_G(P)| = \frac{|G(p)|}{p-1}.$$

Therefore Theorem 3.2 of [8] implies that Moretó's conjecture holds for sporadic simple groups.

We also conclude that this conjecture is valid for alternating groups A_n except $n = 8, 10$ by the following result of J. X. Bi in [3]. For a group G, Bi proved that if $|G| = |A_n|$ and $N_G(R)$ and $N_{A_n}(S)$ have the same order, where R and S are the Sylow p-subgroups of G and A_n, respectively, with p the largest prime not exceeding n and $n \neq 8, 10$, then $G \cong A_n$ (see [3, Theorem 1.2]). We prove the following theorem.

Theorem 1.6. *Suppose that a group G fulfills the conditions of Conjecture 1.2 with $S = A_n$ for $n \geq 5$. Then*

(1) *If $n \neq 8, 10$, then $G \cong S$.*
(2) *If $S = A_8$, then $G \cong A_8$ or $L_3(4)$.*
(3) *If $S = A_{10}$, then $G \cong A_{10}$ or $G \cong J_2 \times Z(G)$, where $Z(G)$ is a cyclic group of order 3.*

Note that in Conjecture 1.2, for a given simple group S, the largest prime p in $\pi(S)$ and the number of elements of order p in S play an important role in recognition of S. On the other hand, almost 40 years ago, M. Herzog in [7] investigated the influence of the number of involutions on the structure of simple groups and proved that many classes of simple groups are characterized by the numbers of their involutions. Furthermore, Herzog conjectured in [7] that if two simple groups have the same number of involutions, then they have the same order. However, this conjecture is not true in general and in [21], M. Zarrin provided a counterexample as follows. Let $G = L_3(4)$ and $S = S_4(3)$. Then, with notation as above, we have $|G(2)| = |S(2)| = 315$, i.e, both $L_3(4)$ and $S_4(3)$ have 315 elements of order 2. Zarrin also in [21] put forward the following conjecture: If S is a nonabelian simple group and G is a group such that $|G(2)| = |S(2)|$ and $|G(p)| = |S(p)|$ for some odd prime divisor p, then $|G| = |S|$. Later on, Zarrin's conjecture and its related topics were discussed in C. S. Anabanti [1] and I. A. Malinowska [9], respectively. In [1], Anabanti disproved Zarrin's conjecture by the following counterexample. Let $G = L_4(3)$ and $S = L_3(9)$. Then $|G(2)| = |S(2)|$ and $|G(13)| = |S(13)|$, but it is clear that $|G| \neq |S|$. More recently, Anabanti et al in [2] provided infinitely many counterexamples to the conjecture of Herzog mentioned above in the foregoing argements and moreover fourteen new counterexamples to Zarrin's conjecture were also given.

Now it is natural to ask whether or not any nonabelian simple group can be determined by the conditions in Moretó's conjecture together with the number of involutions. However, this does not hold in general. For example, take $G = A_8$ and $S = L_3(4)$, which are not isomorphic. Then $|G| = |S|$, $|G(2)| = |S(2)|$ and $|G(7)| = |S(7)|$, where 7 is the largest prime divisor dividing the orders of G and S. For other counterexamples mentioned above, we have the following result.

Proposition 1.7. *Let $S \in \{L_4(3), L_3(9), O_7(3), S_6(3)\}$ and G be a group such that $|G| = |S|$ and $|G(p)| = |S(p)|$ with p the largest prime dividing the order of G. Then*

(1) *If $S = L_4(3)$, then $G \cong S$.*

(2) *If $S \in \{O_7(3), S_6(3)\}$, then $G \in \{O_7(3), S_6(3)\}$. Furthermore, if $|G(2)| = |S(2)|$ is required, then $G \cong S$.*

(3) *If $S = L_3(9)$, then $G \cong S$ or $L_4(3) \times Z(G)$, where $Z(G)$ is a cyclic group of order 7. In addition, if $G \cong L_4(3) \times Z(G)$, then we also have $|G(2)| = |S(2)|$.*

Suggested by this as well as Theorem 1.6 and Anabanti's result in [1], we have the following conjecture, which strengthens the conditions of Moretó's conjecture.

Conjecture 1.8. *Let G be a group and S be a finite simple group. Then $G \cong S$ if and only if $|G| = |S|$ and for every prime $p \in \pi(G) = \pi(S)$, G and S have the same number of elements of order p .*

2. Preliminary

In this section, we collect some elementary facts which are useful in our proof.

For a group G, define its prime graph (or the Gruenberg–Kegel graph) $\Gamma(G)$ as follows: the vertices are the primes dividing the order of G, two vertices p and q are adjacent if and only if G contains an element of order pq (see [19]). Denote the connected components of the prime graph of G by $T(G) = \{\pi_i(G) | 1 \leqslant i \leqslant t(G)\}$, where $t(G)$ is the number of the prime graph connected components of G. If the order of G is even, assume that the prime 2 is always contained in $\pi_1(G)$. A simple group whose order has exactly n distinct prime divisors is called a simple K_n-group. In addition, for a group G, we call G a 2-Frobenius group if G has a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H respectively. For other notation and terminologies mentioned in this paper, the reader is referred to [6] if necessary.

Lemma 2.1. *Let G be a group with more than one prime graph connected components. Then G is one of the following:*

(i) *a Frobenius or 2-Frobenius group;*

(ii) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent π_1-group, K/H is a nonabelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of* $\mathrm{Out}(K/H)$. *Besides, for $i \geq 2$, $\pi_i(G)$ is also a connected component of $\Gamma(K/H)$.*

Proof. It follows straightforward from Theorem A, Lemmas 2-3 and Proposition 1 in [19].

Lemma 2.2. *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and a Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$ and G has one of the following structures*:

(i) $2 \in \pi(H)$ *and all Sylow subgroups of K are cyclic*;

(ii) $2 \in \pi(K)$, *H is an abelian group, K is a solvable group, the Sylow subgroups of K of odd orders are cyclic groups and the Sylow 2-subgroups of K are cyclic or generalized quaternion groups*;

(iii) $2 \in \pi(K)$, *H is an abelian group, and there exists a normal subgroup K_0 in K such that $|K : K_0| \leq 2$ and $K_0 = M \times SL(2,5)$, where M is a group such that $(|M|, 2 \times 3 \times 5) = 1$ and all Sylow subgroups of M are cyclic.*

Proof. This is Lemma 1.6 in [5].

Lemma 2.3. *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, the order of G/K divides the order of* $\mathrm{Aut}(K/H)$*, and both G/K and K/H are cyclic. Especially, $|G/K| < |K/H|$ and G is soluble.*

Proof. This is Lemma 1.7 in [5].

The following lemma is well-known (see [11, Theorem 3.3.20]).

Lemma 2.4. *Let $R = R_1 \times \cdots \times R_k$, where R_i is a direct product of n_i isomorphic copies of a nonabelian simple group H_i, and H_i and H_j are not isomorphic if $i \neq j$. Then*

$$\mathrm{Aut}(R) \cong \mathrm{Aut}(R_1) \times \cdots \times \mathrm{Aut}(R_k) \text{ and } \mathrm{Aut}(R_i) \cong (\mathrm{Aut}(H_i)) \wr S_{n_i}.$$

Moreover,

$$\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \cdots \times \mathrm{Out}(R_k) \text{ and } \mathrm{Out}(R_i) \cong (\mathrm{Out}(R_i)) \wr S_{n_i}.$$

3. Proof of Main Results

Now we first proceed to prove Theorem 1.6.

Proof of Theorem 1.6. (1) Let G be a group satisfying the hypothesis and p be the largest prime dividing the order of A_n with $n \neq 8, 10$. Then $|G|_p = |A_n|_p = p$ and $|G(p)| = |A_n(p)|$. Therefore G and A_n have the same number of Sylow p-subgroups and so $G \cong A_n$ by Bi's result in [3].

(2) Suppose that $|G| = |A_8| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and $|G(7)| = |A_8(7)| = 2^7 \cdot 3^2 \cdot 5$. Let P be a Sylow 7-subgroup of G. Then $N_G(P)$ is of order $3 \cdot 7$. If $N_G(P) = C_G(P)$, then G has a normal 7-complement L. Then it is easy to see that G has a Hall $\{5, 7\}$-subgroup QP, where Q is a Sylow 5-subgroup of L. Thus, $Q \leq N_G(P)$, a contradiction. Hence, $C_G(P) = P$, which implies that $\Gamma(G)$ is not connected and $\{7\}$ is a connected component of $\Gamma(G)$. Suppose that G is a Frobenius group. Let A and B be the Frobenius kernel and a Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(A), \pi(B)\}$ by Lemma 2.2. Hence $\pi(A) = \{7\}$ or $\pi(B) = \{7\}$. Since $|A| - 1$ is divisible by $|B|$, both cases can not occur by the order of G. If G is a 2-Frobenius group, then G is solvable by Lemma 2.3 and so G has a Hall $\{5, 7\}$-subgroup, which is impossible by the preceding arguments. Hence G is neither a Frobenius group nor a 2-Frobenius group. By Lemmas 2.1, we conclude that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent π_1-group, K/H is a nonabelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of $\mathrm{Out}(K/H)$. Besides, for $i \geq 2$, $\pi_i(G)$ is also a connected component of $\Gamma(K/H)$. In particular, 7 is a connected component of $\Gamma(K/H)$. By [22, Table 1], we see that K/H is isomorphic to one of the following groups:

$$A_8, L_3(4), A_7, L_2(8), L_2(7).$$

Suppose that $K/H \cong A_7$. Then $|K/H| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$ and therefore $|H| = 2^2$ or 2^3 since $|G/K|$ divides the order of $\mathrm{Out}(K/H)$. Let N be a minimal normal subgroup of G contained in H. If $|N| \leq 2^2$, then P acts trivially on N, a contradiction. Hence we may assume that $N = H$ is elementary abelian of order 2^3 and it follows that $G = K$. If $C_G(H) = H$, then K/H is isomorphic to a subgroup of $GL(3, 2)$, which is impossible. If $C_G(H) = G$, then $H \leq N_G(P)$, a contradiction. Hence K/H can not be isomorphic to A_7.

If $K/H \cong L_2(8)$ or $L_2(7)$, then 5 does not divide the order of $\mathrm{Aut}(K/H)$ by [6] and so 5 divide the order of H. Let H_5 be the Sylow 5-subgroup of H. Then $H_5 \leq C_G(P)$ as above.

Therefore $K/H \cong A_8$ or $L_3(4)$ and consequently $G \cong A_8$ or $L_3(4)$.

(3) Suppose that $|G| = |A_{10}|$ and $|G(7)| = |A_{10}(7)| = 2^7 \cdot 3^3 \cdot 5^2$. Then $|N_G(P)| = 2 \cdot 3^2 \cdot 7$ with P a Sylow 7-subgroup of G.

Let K be the largest normal solvable subgroup of G. Then $G \neq K$. Otherwise, if G is a solvable group, then G has a Hall subgroup QP of order $5^2 \cdot 7$, where Q is a Sylow 5-subgroup of G. Pick a minimal normal subgroup N of QP. If $N = P$, then $Q \leq N_G(P)$, which is impossible. If N is a cyclic group of order 5 or an elementary abelian group of order 25, then P acts trivially on N and therefore $N \leq N_G(P)$, another contradiction. Hence K is a proper subgroup of G.

Furthermore, we assert that $7 \notin \pi(K)$. If $7 \in \pi(K)$, then P is a Sylow 7-subgroup of K and so $G = N_G(P)K$. By the order of $N_G(P)$, $N_G(P)$ is a solvable group, which implies that G is also solvable, a contradiction. Hence $7 \notin \pi(K)$. Let N be a minimal normal subgroup of G/K. Then N is a direct product of nonabelian simple groups and without loss of generality, we can assume that $7 \in \pi(N)$. Otherwise, if 7 does not divide the order of any minimal normal subgroup of G/K, then 2^{14} divide $|G|$ by Lemma 2.4, a contradiction by our hypothesis. Moreover we have that N is a simple group. By [22, Table 1], we see that N is isomorphic to one of the following groups:

$$A_{10}, J_2, A_8, L_3(4), A_7, U_3(3), L_2(8), L_2(7).$$

Since $N_{G/K}(PK/K) = N_G(P)K/K$, we see that N is the unique minimal normal subgroup of G/K. Otherwise, $|N_{G/K}(PK/K)| \geq 2^2 \cdot 3 \cdot 5 \cdot 7$, a contradiction. Let $N = H/K$. Then we have that $|G/H|$ divides the order of $\mathrm{Out}(N)$.

If $N \cong A_8$, then 5 divides the order of K and so one can deduce that 5 divides $|N_G(P)|$, a contradiction.

If $N \cong L_3(4)$, then 5 divides the order of K since $|\mathrm{Out}(L_3(4))| = 12$ and so we have a contradiction.

If $N \cong A_7, U_3(3), L_2(8), L_2(7)$, we also have that 5 divides $|K|$. Hence we obtain that N is J_2 or A_{10}. Suppose that $N \cong J_2$. Then $|N| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ and $G/K = N$ since $|\mathrm{Out}(J_2)| = 2$. It follows that $|K| = 3$. Clearly, $C_G(K) > K$, which forces $C_G(K) = G$ and so $K = Z(G)$. It is obvious that $G' \cap Z(G) = Z(G)$ or 1. But the Schur multiplier of J_2 is a cyclic group of order 2. Hence $G' \cap Z(G) = 1$ and so $G \cong J_2 \times Z(G)$, where $Z(G)$ is a cyclic group of order 3. By [6], we get that $|J_2(7)| = 2^7 \cdot 3^3 \cdot 5^2$ and so $|G(7)| = 2^7 \cdot 3^3 \cdot 5^2$, as wanted. At last, if $N \cong A_{10}$, then $G \cong A_{10}$ as well.

Thus, the proof is complete. $\square$

Proof of Proposition 1.7. (1) If $S = L_4(3)$, we have $|G| = 2^7 \cdot 3^6 \cdot 5 \cdot 13$ and $|G(13)| = 2^9 \cdot 3^6 \cdot 5$. Let P be a Sylow 13-subgroup of G. Then $N_G(P)$ is of order $3 \cdot 13$. Similarly as in the case (2) of the proof of Theorem 1.6, one can show that $G \cong S$.

(2) Assume that $S = O_7(3)$ or $S_6(3)$. Then by the hypothesis and [6], $|G| = 2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$ and $|G(13)| = 2^{10} \cdot 3^9 \cdot 5 \cdot 7$. Let P be a Sylow 13-subgroup of G and K be the largest normal solvable subgroup of G. Then it is clear that $G \neq K$. Note that $|N_G(P)| = 2 \cdot 3 \cdot 13$. Then $N_G(P)$ is solvable, which implies that $13 \notin \pi(K)$. Hence we have $N_{G/K}(PK/K) = N_G(P)K/K$. Let L/K be a minimal normal subgroup of G/K. By Lemma 2.4, we may assume that $13 \in \pi(L/K)$ and so L/K is a nonabelian simple group. We may further assume that L/K is the unique minimal normal subgroup of G/K since $|N_G(P)| = 2 \cdot 3 \cdot 13$. By [22, Table 1], we have that L/K is isomorphic to one of the following groups:

$$O_7(3), S_6(3), L_3(9), L_2(64), Sz(8), G_2(3), L_2(27), L_2(13), L_4(3), L_3(3).$$

If $L/K \cong L_3(9)$, then the order of $N_{G/K}(PK/K)$ is divided by 7 in view of [6], a contradiction.

If $L/K \cong L_2(64)$, then L/K has a cyclic subgroup of order $5 \cdot 13$, which implies that 5 divides the order of $N_{G/K}(PK/K)$, a contradiction.

If $L/K \cong Sz(8)$, then by [6], we have that PK/K is normalized by a group of order 4, a contradiction.

If $L/K \cong G_2(3), L_2(27), L_2(13), L_4(3)$ or $L_3(3)$, then by [6], we see that $\pi(K)$ contains 5 or 7, since the order of $\mathrm{Out}(L/K)$ is not divisible by 5 or 7. It follows that P is normalized by a cyclic group of order 5 or 7, a contradiction.

Hence we obtain that $L/K \cong O_7(3)$ or $S_6(3)$ and so $G \cong O_7(3)$ or $S_6(3)$.

Finally, since the number of involutions in $O_7(3)$ is not equal to the number of involutions in $S_6(3)$, we see that the second statement holds.

(3) By the hypothesis and [6], we have that $|G| = 2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ and $|G(13)| = 2^9 \cdot 3^6 \cdot 5$. Let P be a Sylow 13-subgroup of G. Then we have $|N_G(P)| = 3 \cdot 7 \cdot 13$. Let K be the largest normal solvable subgroup of G. Similarly as above, we conclude that G/K has a chief factor L/K such that $L/K \cong L_3(9)$ or $L_4(3)$. If $L/K \cong L_3(9)$, then the result follows. If $L/K \cong L_4(3)$, then we have that K is cyclic of order 7 since $|\mathrm{Out}(L_4(3))| = 2$. Therefore $G/K = L/K \cong L_4(3)$ since $|L_4(3)| = 2^7 \cdot 3^6 \cdot 5 \cdot 13$. Similarly as in the proof of Theorem 1.6, we obtain that $G \cong L_4(3) \times Z(G)$, where $Z(G)$ is cyclic of order 7. In this case, we also have $|G(2)| = |S(2)|$ by [6]. $\square$

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