Trudy Instituta Matematiki i Mekhaniki UrO RAN

Vol. 27 No. 3 2021

COOPERATIVE DIFFERENTIAL GAMES WITH PARTNER SETS ON NETWORKS[1]

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In the paper, the differential games on networks with partner sets are considered. The payoffs of a given player depend on his actions and the actions of the players from his partner set. The cooperative version of the game is proposed, and the special type of characteristic function is introduced. It is proved the constructed cooperative game is convex. Using the properties of the payoff functions and the constructed characteristic function, the Shapley Value and τ-value are computed. It is also proved that in this special class of differential games the Shapley value is time-consistent.

Keywords: Shapley value, differential network game, time consistency, partner sets.

Л. А. Петросян, Д. Янг, Я. Б. Панкратова. Кооперативные дифференциальные игры с партнерскими множествами на сетях.

Рассматриваются сетевые дифференциальные игры с партнерскими множествами. Выигрыш каждого игрока зависит от его действий и действий игроков из его партнерского множества. В статье предложена кооперативная версия игры и введен особый тип характеристической функции. Доказано, что построенная кооперативная игра является выпуклой. Свойства функций выигрыша и построенной характеристической функции используются для вычисления вектора Шепли и τ-вектора. Также доказано, что в указанном классе дифференциальных игр вектор Шепли динамически устойчив.

Ключевые слова: вектор Шепли, сетевая дифференциальная игра, динамическая устойчивость, партнерские множества.

MSC: 91A23, 91A12, 91A43

DOI: 10.21538/0134-4889-2021-27-3-286-295

Introduction

Differential games on networks are a relatively new class of differential games (Isaacs (1965)[5], Krasovsky and Subbotin (1974)[6]). It is necessary to mention the papers of Wie (1993)[14], Pai (2010) [8], Zhang at al. (2018)[19], Meza and Lopez-Barrientos (2016)[7], Petrosyan (2010)[9], Gao and Pankratova (2017)[3], and the paper of Petrosyan and Yeung (2020)[11] where the new characteristic function in differential cooperative network game was introduced in a special case when the payoffs of players depend only upon their actions and actions of neighbours in the network. In this paper, we consider the case when a player's payoff depends upon payoffs of players from his partner set. The rules of the game allow that a player may belong to many partner sets. When constructing characteristic function, we suppose that left out players can cut connections with those who decide to form a coalition. This simplifies the computation of characteristic function, the Shapley Value and τ-value. It is shown that the corresponding cooperative game is convex.

1. Formulation of a Class of Differential Network Games

Consider a class of n-person differential games on network with game horizon $[t_0, T]$. The players are connected in a network system. We use $N = \{1, 2, \ldots, n\}$ to denote the set of players in the

[1]The first author is supported by Russian Science Foundation, the grant “Optimal Behavior in Conflict-Controlled Systems” (project no. 17-11-01079).

network. The nodes of the network are used to represent the players from the set N. We also denote the set of nodes by N and denote the set of all arcs in network N by L. The arcs in L are the $arc\ (i,j) \in L$ for players $i, j \in N$, $i \neq j$. For notational convenience, we denote the set of players connected to player i as $\tilde{K}(i) = \{j : arc(i,j) \in L\}$, for $i \in N$.

We suppose also that a family of subsets $M_1, \ldots, M_k, \ldots, M_l$, $M_k \subset N$, $k = 1, \ldots, l$, of the set N is given. It is supposed that $|M_k| \geq 2$, and for all $i \in N$ there exist $l \in N$, such that $i \in M_l$. Also for each two nodes $z_1 \in N \cap M_k$, $z_2 \in N \cap M_k$ there exist a path connecting z_1 and z_2 in M_k. The sets $M_1, \ldots, M_k, \ldots, M_l$ are called "partner" sets.

Let $x^i(\tau) \in \mathbb{R}^m$ be the state variable of player $i \in N$ at time τ, and $u^i(\tau) \in U^i \subset \mathbb{R}^k$ the control variable of player $i \in N$.

Every player $i \in N$ can cut the connection with any other player from the set M_k at any instant of time.

The state dynamics of the game is

$$\dot{x}^i(\tau) = f^i(x^i(\tau), u^i(\tau)), \quad x^i(t_0) = x^i_0, \quad \text{for } \tau \in [t_0, T] \text{ and } i \in N. \tag{1.1}$$

The function $f^i(x^i, u^i)$ is continuously differentiable in x^i and u^i.

The payoff function of player i depends upon his state variable, his own control variable and the state variables of players from the sets M_k to which he belongs.

In particular, the payoff of player i is given as

$$H_i(x^1_0, \ldots, x^n_0, u^1, \ldots, u^n) = \sum_{k=1}^{l} \left(\sum_{\substack{j \in M_k, \\ M_k \ni i}} \int_{t_0}^{T} h^j_{ik}(x^i(\tau), x^j(\tau)) d\tau \right). \tag{1.2}$$

The term $h^j_{ik}(x^i(\tau), x^j(\tau))$ is the instantaneous gain that player i can obtain through network links with player $j \in M_k$, $M_k \ni i$ (note that the pair $(i,i) \notin L$). The functions $h^j_{ik}(x^i(\tau), x^j(\tau))$, for $j \in M_k$ are non-negative. For notational convenience, we use $x(t)$ to denote the vector $(x^1(t), x^2(t), \cdots, x^n(t))$.

From formula (1.2) we can see that the payoff of player i is computed as a sum of payoffs which he gets interacting with players $j \in M_k$ ($M_k \ni i$) for all subsets M_k containing player $i \in N$.

Since the set N is finite the sum in (1.2) contains a finite number of summands $\leq |N|$.

Cooperation and Characteristic Function. To achieve group optimality, the players maximize their joint payoff

$$\sum_{i \in N} \sum_{k=1}^{l} \left(\sum_{\substack{j \in M_k, \\ M_k \ni i}} \int_{t_0}^{T} h^j_{ik}(x^i(\tau), x^j(\tau)) d\tau \right) \tag{1.3}$$

subject to dynamics (1.1).

We use $\bar{x}(t) = (\bar{x}^1(t), \bar{x}^2(t), \cdots, \bar{x}^n(t))$ to denote the optimal cooperative trajectory of problem of maximizing (1.3) subject to (1.1). We let the corresponding optimal cooperative trajectory of player i be denoted by $\bar{x}^i(t)$, for $t \in [t_0, T]$ and $i \in N$. The maximized joint cooperative payoff involving all players can then be expressed as

$$\sum_{i \in N} \sum_{k=1}^{l} \left(\sum_{\substack{j \in M_k, \\ M_k \ni i}} \int_{t_0}^{T} h^j_{ik}(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau \right) = \max_{u^1, u^2, \cdots, u^n} \sum_{i \in N} \sum_{k=1}^{l} \left(\sum_{\substack{j \in M_k, \\ M_k \ni i}} \int_{t_0}^{T} h^j_{ik}(x^i(\tau), x^j(\tau)) d\tau \right)$$

subject to dynamics (1.1)

Next, we consider distributing the cooperative payoff to the participating players under an agreeable scheme. Given that the contributions of an individual player to the joint payoff through linked players can be diverse, the Shapley (1953) [12] value provides one of the best solutions in attributing a fair gain to each player in a complex network. One of the contentious issues in using the Shapley value is the determination of the worth of subsets of players (characteristic function).

In this section, we present a new formulation of the worth of coalition $S \subset N$ (Bulgakova and Petrosyan (2019) [1]). In computing the values of characteristic function for coalitions, we evaluate contributions of the players in the process of cooperation and maintain the cooperative strategies for all players along the cooperative trajectory. In particular, we evaluate the worth of the coalitions along the cooperative trajectory as

$$V(S; x_0, T-t_0) = \sum_{i\in S} \sum_{k=1}^{l} \Bigg(\sum_{\substack{j\in M_k \cap S,\\ M_k \ni i}} \int_{t_0}^{T} h_{ik}^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau \Bigg). \tag{1.4}$$

Note that the worth of coalition S is measured by the sum of payoffs of the players in the coalition in the cooperation process with the exclusion of the gains from players outside coalition S. Thus, the characteristic function reflecting the worth of coalition S in (1.4) is formulated along the cooperative trajectory $\bar{x}(t)$.

Similarly, the characteristic function at time $t \in [t_0, T]$ can be evaluated as

$$V(S; \bar{x}(t), T-t) = \sum_{i\in S} \sum_{k=1}^{l} \Bigg(\sum_{\substack{j\in M_k \cap S,\\ M_k \ni i}} \int_{t}^{T} h_{ik}^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau \Bigg). \tag{1.5}$$

For simplicity in notation, we denote the gain that player i can obtain through the network link with player $j \in M_k$ as

$$\alpha_{ij}^k(\bar{x}(t), T-t) = \int_{t}^{T} h_{ik}^j(\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau, \text{ for } t \in [t_0, T]. \tag{1.6}$$

Using the notations in (1.6), we can express the worth of coalition S in (1.4) in the start of the cooperation scheme as

$$V(S; x_0, T-t_0) = \sum_{i\in S} \Bigg[\sum_{k=1}^{l} \Bigg(\sum_{\substack{j\in M_k \cap S,\\ M_k \ni i}} \alpha_{ij}^k(x_0, T-t_0) \Bigg) \Bigg], \tag{1.7}$$

and the worth of coalition S along the cooperative trajectory $\bar{x}(t)$ as

$$V(S; \bar{x}(t), T-t) = \sum_{i\in S} \Bigg[\sum_{k=1}^{l} \Bigg(\sum_{\substack{j\in M_k \cap S,\\ M_k \ni i}} \alpha_{ij}^k(\bar{x}(t), T-t) \Bigg) \Bigg], \quad \text{for } t \in [t_0, T]. \tag{1.8}$$

An important property of the above characteristic function as a measure of the worth of coalition is given below.

Proposition 1. *The characteristic function defined by* (1.7) *and* (1.8) *is convex.*

Proof. Proof the following inequalities:

$$V(S_1 \cup S_2; x_0, T-t_0) \geq V(S_1; x_0, T-t_0) + V(S_2; x_0, T-t_0) - V(S_1 \cap S_2; x_0, T-t_0).$$

Using (1.7), we have

$$V(S_1 \cup S_2; x_0, T-t_0) = \sum_{\substack{i \in S_1 \cup S_2,\\ M_k \ni i}} \sum_{k=1}^{l} \Big(\sum_{j \in M_k \cap (S_1 \cup S_2)} \alpha_{ij}^k (x_0, T-t_0) \Big)$$

$$= \sum_{i \in S_1} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S_1,\\ M_k \ni i}} \alpha_{ij}^k (x_0, T-t_0) \Big) + \sum_{i \in S_2} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S_2,\\ M_k \ni i}} \alpha_{ij}^k (x_0, T-t_0) \Big)$$

$$- \sum_{i \in S_1 \cap S_2} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap (S_1 \cap S_2),\\ M_k \ni i}} \alpha_{ij}^k (x_0, T-t_0) \Big)$$

$$+ \sum_{i \in S_1} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S_2,\\ M_k \ni i}} \alpha_{ij}^k (x_0, T-t_0) \Big) + \sum_{i \in S_2} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S_1,\\ M_k \ni i}} \alpha_{ij}^k (x_0, T-t_0) \Big)$$

$$\geq \sum_{i \in S_1} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S_1,\\ M_k \ni i}} \alpha_{ij}^k (x_0, T-t_0) \Big) + \sum_{i \in S_2} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S_2,\\ M_k \ni i}} \alpha_{ij}^k (x_0, T-t_0) \Big)$$

$$- \sum_{i \in S_1 \cap S_2} \sum_{k=1}^{l} \Big(\sum_{j \in M_k \cap (S_1 \cap S_2), M_k \ni i} \alpha_{ij}^k (x_0, T-t_0) \Big)$$

Hence Proposition 1 follows. □

Similarly, along the cooperative trajectory $\bar{x}(t)$, the following inequalities hold

$$V(S_1 \cup S_2; \bar{x}(t), T-t) \geq V(S_1; \bar{x}(t), T-t) + V(S_2; \bar{x}(t), T-t) - V(S_1 \cap S_2; \bar{x}(t), T-t), \text{ for } t \in [t_0, T].$$

The inequalities in Proposition 1 implies that the game is convex, and so are the subgames along the cooperative trajectory. This also means that the core of the game is not void, and the Shapley value belongs to the core.

From (1.4), (1.5) we get

$$V(S; x_0, T-t_0) = \sum_{i \in S} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S,\\ M_k \ni i}} \int_{t_0}^{t} h_{ik}^j (\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau \Big)$$

$$+ \sum_{i \in S} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S,\\ M_k \ni i}} \int_{t}^{T} h_{ij}^j (\bar{x}(\tau), \bar{x}^j(\tau)) d\tau \Big)$$

$$= \sum_{i \in S} \sum_{k=1}^{l} \Big(\sum_{\substack{j \in M_k \cap S,\\ M_k \ni i}} \int_{t_0}^{t} h_{ik}^j (\bar{x}^i(\tau), \bar{x}^j(\tau)) d\tau \Big) + V(S; \bar{x}(t), T-t) \tag{1.9}$$

The equation (1.9) can be interpreted as time-consistency property of newly introduced characteristic function.

It is necessary to mention that this property of the characteristic function has not been shared by existing characteristic functions in differential games. As we can see in our case the worth of coalitions is measured under the process of cooperation instead of under min-max confrontation (Cao at al. (1963)[2]) or Nash non-cooperative stance. And, any individual player or coalition attempting to act independently will have the links to other players in the network being cut off.

Because of this players outside S in worst case will cut connection with players from S, and players from S will get positive payoffs only interacting with other players from S.

2. Dynamic Shapley Value and τ-value

In this section, we develop a dynamic Shapley value imputation using the defined characteristic function.

Now, we consider allocating the grand coalition cooperative network gain $V(N; x_0, T-t_0)$ to individual players according to the Shapley value imputation. Player i's payoff under cooperation would become

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S\subset N,\\ S\ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \cdot \big[V(S; x_0, T-t_0) - V(S\backslash\{i\}; x_0, T-t_0)\big], \text{ for } i \in N.$$

Invoking (1.7), in our case, we can obtain the cooperative payoff of player i under the Shapley value as

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \cdot \Bigg\{ \sum_{l\in S}\sum_{k=1}^{l}\Bigg(\sum_{\substack{j\in M_k\cap S,\\ M_k\ni i}} \int_{t_0}^{T} h_{ik}^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau\Bigg)$$

$$- \sum_{l\in S\backslash\{i\}}\sum_{k=1}^{l}\Bigg(\sum_{\substack{j\in M_k\cap S\backslash\{i\},\\ M_k\ni i}} \int_{t_0}^{T} h_{ik}^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau\Bigg)\Bigg\}. \tag{2.1}$$

However, in a dynamic framework, the agreed upon optimality principle for sharing the gain has to be maintained throughout the cooperation duration (see Yeung and Petrosyan (2004 and 2016) [16; 17]) for a dynamically consistent solution. Applying the Shapley value imputation in (2.2) to any time instance $t \in [t_0, T]$, we obtain:

$$Sh_i(\bar{x}(t), T-t) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \cdot \Bigg\{ \sum_{l\in S}\sum_{k=1}^{l}\Bigg(\sum_{\substack{j\in M_k\cap S,\\ M_k\ni i}} \int_{t}^{T} h_{ik}^j(\bar{x}^j(\tau), \bar{x}^j(\tau))d\tau\Bigg)$$

$$- \sum_{l\in S\backslash\{i\}}\sum_{k=1}^{l}\Bigg(\sum_{\substack{j\in M_k\cap S\backslash\{i\},\\ M_k\ni i}} \int_{t}^{T} h_{ik}^j(\bar{x}^j(\tau), \bar{x}^j(\tau))d\tau\Bigg)\Bigg\} \tag{2.2}$$

The Shapley value imputation in (2.1), (2.2) is based on characteristic function evaluates along the optimal cooperative trajectory and it attributes the contributions of the players under the optimal cooperation process. Indeed, it can be regarded as optimal trajectory dynamic Shapley value. In addition, this Shapley value imputation (2.1)–(2.2) fulfils the property of time consistency.

Proposition 2. *The Shapley value imputation in* (2.1), (2.2) *satisfies the time consistency property.*

Proof. By direct computation we get:

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \cdot \Bigg\{ \sum_{l\in S}\sum_{k=1}^{l}\Bigg(\sum_{\substack{j\in M_k\cap S,\\ M_k\ni i}} \int_{t_0}^{t} h_{ik}^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau\Bigg)$$

$$- \sum_{l\in S\backslash\{i\}}\sum_{k=1}^{l}\Bigg(\sum_{\substack{j\in M_k\cap S\backslash\{i\},\\ M_k\ni i}} \int_{t_0}^{t} h_{ik}^j(\bar{x}^i(\tau), \bar{x}^j(\tau))d\tau\Bigg)\Bigg\} + Sh_i(\bar{x}(t), T-t) = \ i \in N,$$

which exhibits the time consistency property of the Shapley value imputation $Sh_i(\bar{x}(t), T-t)$, for $t \in [t_0, T]$. □

This is the first time that a Shapley value measure itself in a dynamic framework fulfils the property of time consistency (see existing dynamic Shapley value measures which do not share this property in Gromova (2016) [4], Petrosyan and Zaccour (2003) [10], Yeung (2010) [15], Yeung and Petrsoyan (2016 and 2018)) [17; 18]. Using this Shapley value formulation, the cooperative game solution would automatically satisfy the condition of cooperative time consistency (see Yeung and Petrosyan (2004 and 2016) [16; 17]). Crucial to the analysis is the design of an Imputation Distribution Procedure (IDP) such that the Shapley imputation (2.1), (2.2) can be realized.

An IDP leading to the realization of the Shapley imputation $Sh_i(x(t), T-t)$ in (2.1), (2.2) has to satisfy

$$\int_{t_0}^{T} \beta_i(\tau) d\tau = Sh_i(x_0, T-t_0) \text{ and } \int_{t}^{T} \beta_i(\tau) d\tau = Sh_i(\bar{x}(t), T-t).$$

Following Yeung and Petrosyan (2004 and 2016), we obtain

Proposition 3. *An imputation distribution procedure* (IDP) *giving player $i \in N$ at time $t \in [t_0, T]$ an allotment*

$$\beta_i(t) = -\frac{d}{dt} Sh_i(\bar{x}(t), T-t) = \sum_{\substack{S \subset N \\ S \ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \cdot \left\{ \sum_{l \in S} \sum_{k=1}^{l} \Bigg(\sum_{\substack{j \in M_k \cap S, \\ M_k \ni i}} h_{ik}^j(\bar{x}^l(t), \bar{x}^j(t)) \Bigg) \right.$$

$$\left. - \sum_{l \in S \backslash \{i\}} \sum_{k=1}^{l} \Bigg(\sum_{\substack{j \in M_k \cap S \backslash \{i\}, \\ M_k \ni i}} h_{ik}^j(\bar{x}^l(t), \bar{x}^j(t)) \Bigg) \right\}$$

would lead to the realization of the Shapley value imputation.

Consider now allocating the grand coalition cooperative network gain $V(N; x_0, T-t_0)$ to individual players according to the τ-value imputation [13]. In our case, the τ-value can be computed according to (2.3) and (2.4)

$$\tau_i(x_0, T-t_0) = \lambda\big(V(N; x_0, T-t_0) - V(N \backslash \{i\}; x_0, T-t_0)\big) + (1-\lambda) V\big(\{i\}; x_0, T-t_0\big) \tag{2.3}$$

where λ is satisfies

$$\sum_{i \in N} \Big(\lambda\big(V(N; x_0, T-t_0) - V(N \backslash \{j\}; x_0, T-t_0)\big) + (1-\lambda) V(\{i\}; x_0, T-t_0) \Big) = V(N; x_0, T-t_0). \tag{2.4}$$

We have

$$V(\{i\}; x_0, T-t_0) = 0, \tag{2.5}$$

since the arc (i, i) is not contained in L.

From (2.4) using (2.5), we get

$$\lambda \sum_{j \in N} \big(V(N; x_0, T-t_0) - V(N \backslash \{j\}; x_0, T-t_0)\big) = V(N; x_0, T-t_0),$$

$$\lambda = \frac{V(N; x_0, T-t_0)}{\sum_{j \in N} \big(V(N; x_0, T-t_0) - V(N \backslash \{j\}; x_0, T-t_0)\big)}.$$

And

$$\tau_i(x_0, T-t_0) = \frac{V(N;x_0,T-t_0) - V(N\backslash\{i\},x_0,T-t_0)}{\sum\limits_{j\in N}\big(V(N;x_0,T-t_0) - V(N\backslash\{j\};x_0,T-t_0)\big)} V(N;x_0,T-t_0).$$

In the subgame along the cooperative trajectory

$$\tau_i(\bar{x}(t), T-t) = \frac{V(N;\bar{x}(t),T-t) - V(N\backslash\{i\};\bar{x}(t),T-t)}{\sum\limits_{j\in N}\big(V(N;\bar{x}(t),T-t) - V(N\backslash\{j\};\bar{x}(t),T-t)\big)} V(N;\bar{x}(t),T-t).$$

The time consistency property of dynamic τ-value can be satisfied if we introduce the imputation distribution procedure as

$$\beta_i(t) = -\frac{d}{dt}\tau_i(\bar{x}(t), T-t).$$

3. Examples

To simplify formulas we will denote $\alpha_{ij}^k(x_0.T-t_0)$ as α_{ij}^k, and introduce following new notation

$$\alpha_{ij}^k + \alpha_{ji}^k = A^k(i,j),$$

wicth is equal to the payoff of the pair (i,j), for $i\in M_k$, $j\in M_k$.

The payoff of the pair (i,j) in the group M_k will be

$$\sum_{j\in M_k,\ M_k\ni i} A^k(i,j) = \sum_{i\in M_k,\ M_k\ni j} A^k(j,i).$$

The total payoff of the pair (i,j) in the game can be expressed as

$$\sum_{k=1}^{n}\Big(\sum_{j\in M_k,\ M_k\ni i} A^k(i,j)\Big) = \sum_{k=1}^{n}\sum_{i\in M_k,\ M_k\ni j} A^k(j,i) = A(i,j) = A(j,i).$$

The total payoff of the pair (i,j) in the coalition S is equal to

$$v(S) = \sum_{i\in S,\ j\in S} A(i,j).$$

Example 1. Consider the following 4 player network game (see Fig. 1): Let $M_1 = \{1,2\}$,

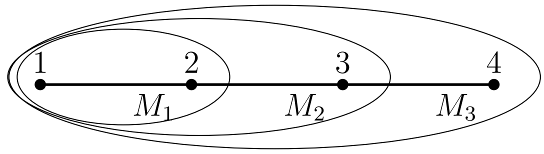

Fig. 1.

$M_2 = \{1,2,3\}$, $M_3 = \{1,2,3,4\}$ be partner sets.

Using above notations, compute the values of characteristic function

$V(\{1,2\}) = A^1(1,2) + A^2(1,2) + A^3(1,2) = A(1,2)$,

$V(\{1,3\}) = A^2(1,3) + A^3(1,3) = A(1,3)$,

$V(\{1,4\}) = A^3(1,4) = A(1,4)$,

$V(\{2,3\}) = A^2(2,3) + A^3(2,4) = A(2,3)$,

$V(\{3,4\}) = A^3(3,4) = A(3,4)$,

$V(\{2,4\}) = A^3(2,4) = A(2,4)$,

$V(\{1,2,3\}) = A(1,2) + A(1,3) + A(2,3) = V(\{N\backslash 4\})$,
$V(\{1,2,4\}) = A(1,2) + A(1,4) + A(2,4) = V(\{N\backslash 3\})$,
$V(\{1,3,4\}) = A(1,3) + A(1,4) + A(3,4) = V(\{N\backslash 2\})$,
$V(\{2,3,4\}) = A(2,3) + A(2,4) + A(3,4) = V(\{N\backslash 1\})$,
$V(N) = A(1,2) + A(1,3) + A(2,4) + A(2,3) + A(2,4) + A(3,4)$.
Using the formula

$$\tau_i = \frac{V(N) - V(N\backslash\{i\})}{\sum\limits_{j\in N}(V(N) - V(N\backslash\{j\}))} V(N),$$

and computing the term

$$\sum_i V(N\backslash\{i\}) = 2(A(2,3) + A(2,4) + A(3,4) + A(1,3) + A(1,4) + A(1,2)) = 2V(N),$$

we get

$$\tau_i = \frac{V(N) - V(N\backslash\{i\})}{4V(N) - 2V(N)} V(N) = \frac{V(N) - V(N\backslash\{i\})}{2},$$

and

$$\tau_1 = \frac{A(1,2) + A(1,3) + A(1,4)}{2}, \quad \tau_2 = \frac{A(1,2) + A(2,3) + A(2,4)}{2},$$

$$\tau_3 = \frac{A(1,3) + A(2,4) + A(1,4)}{2}, \quad \tau_4 = \frac{A(1,4) + A(2,4) + A(3,4)}{2}.$$

Now, compute the Shapley value

$$Sh_1 = \frac{1}{12}(A(1,2) + A(1,3) + A(1,2)) + \frac{1}{12}(A(1,2) + A(1,3) + A(1,2) + A(1,4) + A(1,3) + A(1,4))$$

$$+ \frac{1}{4}(A(1,2) + A(1,3) + A(1,4)) = \frac{A(1,2) + A(1,3) + A(1,4)}{2}.$$

Acting in the same way and calculating the remaining components of the Shapley value, we can see that in this example the τ-value coincides with the Shapley value.

Example 2. Consider the following 4 player network game (see Fig. 2):

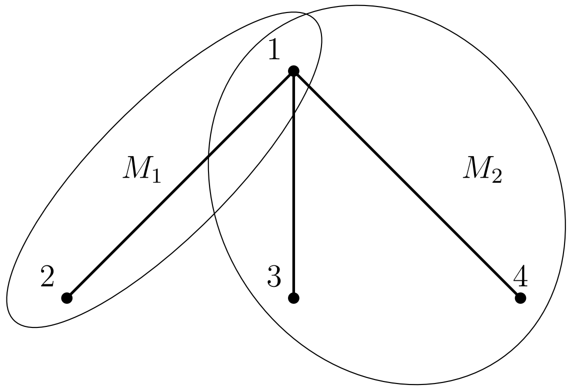

Fig. 2.

Let $M_1 = \{1,2\}$ and $M_2 = \{1,3,4\}$ be partner sets.

For this network structure the characteristic function is defined as

$$
\begin{aligned}
&V(\{1,2\}) = A^1(1,2) = A(1,2),\\
&V(\{1,3\}) = A^2(1,3) = A(1,3),\\
&V(\{1,4\}) = A^2(1,4) = A(1,4),\\
&V(\{2,3\}) = 0,\\
&V(\{2,4\}) = 0,\\
&V(\{3,4\}) = A^2(3,4) = A(3,4),\\
&V(\{1,2,3\}) = A^1(1,2) + A^2(1,3) = A(1,2) + A(1,3) = V(N\backslash\{4\}),\\
&V(\{1,2,4\}) = A^1(1,2) + A^2(1,4) = A(1,2) + A(1,4) = V(N\backslash\{3\}),\\
&V\{(1,3,4\}) = A^2(1,3) + A^2(1,4) + A^2(3,4) = A(1,3) + A(1,4) + A(3,4) = V(N\backslash\{2\}),\\
&V(\{2,3,4\}) = A^2(3,4) = A(3,4) = V(N\backslash\{1\}),\\
&V(\{1,2,3,4\}) = A^1(1,2) + A^2(1,3) + A^2(1,4) + A^2(3,4) = A(1,2) + A(1,3) + A(1,4) + A(3,4).
\end{aligned}
$$

Compute τ-value

$$
\tau_1 = \frac{A(1,2) + A(1,3) + A(1,4)}{2}, \quad \tau_2 = \frac{A(1,2)}{2}, \quad \tau_3 = \frac{A(1,3) + A(3,4)}{2}, \quad \tau_4 = \frac{A(1,4) + A(3,4)}{2}.
$$

Computing the Shapley value we get the Shapley value coincides with τ-value

$$
\tau_i = Sh_i, \quad i = \overline{1,4}.
$$

Conclusion

A special type of differential game on network is considered. The new notion of a partner set is introduced. Two players may belong to a partner set if there is a path connecting them in the network. One player can be a member of different partner sets. It is supposed that player gets positive income resulting from the communication with other players from his partner sets. Using the novel form for measuring the worth of coalitions, in computing the characteristic function, we evaluate contributions of the players in the process of cooperation and define cooperative strategies of players along the cooperative trajectory. The explicit formulas for the Shapley value and τ-value are derived. It is proved that the constructed characteristic function is convex and the main solution concepts based on this characteristic function (the Shapley value, and τ-value) are time-consistent (dynamic stable).

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Received May 31, 2021
Revised June 15, 2021
Accepted August 2, 2021

Funding Agency: The first author is supported by Russian Science Foundation, the grant "Optimal Behavior in Conflict-Controlled Systems" (project no. 17-11-01079).

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Cite this article as: L. A. Petrosyan, D. Yeung, Y. B. Pankratova. Cooperative Differential Games with Partner Sets on Networks. *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2021, vol. 27, no. 3, pp. 286–295 .