Trudy Instituta Matematiki i Mekhaniki UrO RAN

Vol. 27 No. 1 2021

УДК 512.542

## ON THE LATTICES OF THE $\omega$-FIBERED FORMATIONS OF FINITE GROUPS[1]

**S. P. Maksakov**

Only finite groups and classes of finite groups are considered. The lattice approach to the study of formations of groups was first applied by A.N. Skiba in 1986. L.A. Shemetkov and A.N. Skiba established main properties of lattices of local formations and $\omega$-local formations where $\omega$ is a nonempty subset of the set $\mathbb{P}$ of all primes. An $\omega$-local formation is one of types of $\omega$-fibered formations introduced by V.A. Vedernikov and M.M. Sorokina in 1999. Let $f : \omega \cup \{\omega'\} \to$ {formations of groups}, where $f(\omega') \neq \varnothing$, and $\delta : \mathbb{P} \to$ {nonempty Fitting formations} are the functions. Formation $\mathfrak{F} = (G \mid G/O_\omega(G) \in f(\omega')$ and $G/G_{\delta(p)} \in f(p)$ for all $p \in \omega \cap \pi(G))$ is called an $\omega$-fibered formation with a direction $\delta$ and with an $\omega$-satellite $f$, where $O_\omega(G)$ is the largest normal $\omega$-subgroup of the group $G$, $G_{\delta(p)}$ is the $\delta(p)$-radical of the group $G$, i.e. the largest normal subgroup of the group $G$ belonging to the class $\delta(p)$, and $\pi(G)$ is the set of all prime divisors of the order of the group $G$. We study properties of lattices of $\omega$-fibered formations of groups. In this work we have proved the modularity of the lattice $\Theta_{\omega\delta}$ of all $\omega$-fibered formations with the direction $\delta$. Its sublattice $\Theta_{\omega\delta}(\mathfrak{F})$ for the definite $\omega$-fibered formation $\mathfrak{F}$ with the direction $\delta$ is considered. We have established sufficient conditions under which the lattice $\Theta_{\omega\delta}(\mathfrak{F})$ is a distributive lattice with complements.

Keywords: finite group, class of groups, formation, $\omega$-fibered formation, lattice, modular lattice, distributive lattice, lattice with complements.

**С. П. Максаков. О решетках $\omega$-веерных формаций конечных групп.**

Рассматриваются только конечные группы и классы конечных групп. Решеточный подход к изучению формаций групп был впервые применен А.Н. Скибой в 1986 г. Л.А. Шеметков и А.Н. Скиба установили основные свойства решеток локальных формаций и $\omega$-локальных формаций, где $\omega$ — непустое подмножество множества $\mathbb{P}$ всех простых чисел. В 1999 г. В.А.Ведерников и М.М. Сорокина ввели понятие $\omega$-веерных формаций, одним из типов которых являются $\omega$-локальные формации. Рассмотрим функции $f : \omega \cup \{\omega'\} \to$ {формации групп}, где $f(\omega') \neq \varnothing$, и $\delta : \mathbb{P} \to$ {непустые формации Фиттинга}. Формация $\mathfrak{F} = (G \mid G/O_\omega(G) \in f(\omega')$ и $G/G_{\delta(p)} \in f(p)$ для всех $p \in \omega \cap \pi(G))$ называется $\omega$-веерной формацией с направлением $\delta$ и $\omega$-спутником $f$, где $O_\omega(G)$ — наибольшая нормальная $\omega$-подгруппа $G$, $G_{\delta(p)}$ — $\delta(p)$-радикал $G$, т.е. наибольшая нормальная подгруппа $G$ из класса $\delta(p)$, и $\pi(G)$ — множество простых делителей порядка группы $G$. Изучаются свойства решеток $\omega$-веерных формаций групп. Доказана модулярность решетки $\Theta_{\omega\delta}$ всех $\omega$-веерных формаций с направлением $\delta$. Рассмотрена её подрешетка $\Theta_{\omega\delta}(\mathfrak{F})$ для некоторой $\omega$-веерной формации $\mathfrak{F}$ с направлением $\delta$. Найдены достаточные условия, при которых $\Theta_{\omega\delta}(\mathfrak{F})$ является дистрибутивной решеткой с дополнениями.

Ключевые слова: конечная группа, класс групп, формация, $\omega$-веерная формация, решетка, модулярная решетка, дистрибутивная решетка, решетка с дополнениями.

**MSC:** 20D10, 20E17

**DOI**: 10.21538/0134-4889-2021-27-1-258-267

### Введение

The lattice methods play an important role in the study of the properties of many algebraic objects. A partially ordered set is called a lattice if for any two elements there exist an exact lower bound (the lattice intersection) and an exact upper bound (the lattice union). It is well known that the set $S(G)$ of all subgroups of a finite group $G$ is a lattice (for any $A, B \in S(G)$ the lattice intersection of $A$ and $B$ is equal to $A \cap B$ and the lattice union of $A$ and $B$ is equal to the subgroup of $G$ which is generated by the union $A \cup B$); and the set of all normal subgroups of the group $G$ is a sublattice of this lattice. In 1939 H. Wielandt established the fact that the set of all subnormal

[1]This paper is based on the results of the 2020 Ural Workshop on Group Theory and Combinatorics.

subgroups of any finite group also forms the lattice [21]. L.A. Shemetkov and A.N. Skiba in their monograph [9] studied the lattice properties of manifolds and formations of algebraic systems. As the theory of classes of finite groups developed, two problems have become considered. The first problem is connected with the study of the lattice properties of new types of subgroups of finite groups which are defined by given classes. For instance, A.F. Vasil'ev, S.F. Kamornikov, V.N. Semenchuk in [16] obtained the description of a hereditary local formation $\mathfrak{F}$ such that in any finite group $G$ the set of all $\mathfrak{F}$-subnormal subgroups of $G$ is a lattice. The second problem is connected with the study of the lattice properties of the classes of groups (formations, Fitting classes and others). A set of groups is called a class if it contains with any its group $G$ all groups which are isomorphic to $G$. The class $\mathfrak{F}$ is said to be: a formation if $\mathfrak{F}$ is $Q$-closed ($G \in \mathfrak{F}$, $N \lhd G \Rightarrow G/N \in \mathfrak{F}$) and $R_0$-closed ($G/A \in \mathfrak{F}$, $G/B \in \mathfrak{F} \Rightarrow G/(A \cap B) \in \mathfrak{F}$); a Fitting class if $\mathfrak{F}$ is $S_n$-closed ($G \in \mathfrak{F}$, $N \lhd G \Rightarrow N \in \mathfrak{F}$) and $R$-closed ($G = AB$, $A \lhd G$, $B \lhd G$, $A, B \in \mathfrak{F} \Rightarrow G \in \mathfrak{F}$); a Fitting formation if $\mathfrak{F}$ is a formation and a Fitting class. The lattice approach to the study of formations of finite groups was first applied by A.N. Skiba in [12]. L.A. Shemetkov and A.N. Skiba in [9] established important properties of the lattice of all formations of finite groups, in particular, they proved that this lattice is modular but not distributive.

Among the classes of finite groups the central place is occupied by local formations and local Fitting classes introduced respectively by W. Gaschutz [3] and B. Hartley [4]. A.N. Skiba in [13] established the crucial properties of the lattice of all $\tau$-closed $n$-multiple local formations, where $\tau$ is a subgroup functor, $n \in \mathbb{N}$ (for instance, inductance, modularity, algebraicity, etc.), proved its nondistributivity and studied the Boolean sublattices of this lattice. Let $\omega$ be a nonempty set of primes, $\mathfrak{L}$ is a nonempty class of simple groups. In the fundamental works [15] and [14] respectively the basic properties of a lattice of $n$-multiple $\omega$-local formations and of a lattice of $n$-multiple $\mathfrak{L}$-composition formations were established and some problems connected with their further study were formulated. The monograph [20] presents the most complete presentation of the results obtained over the past decades about lattices of $n$-multiple $\omega$-local formations and $n$-multiple $\omega$-local Fitting classes of groups. An $\omega$-local formation (an $\omega$-local Fitting class) is one of the representatives of the series of $\omega$-fibered formations ($\omega$-fibered Fitting classes) of groups introduced by V.A. Vedernikov, M.M. Sorokina in [18] and an $\Omega$-composition formation (an $\Omega$-composition Fitting class) is one of the types of $\Omega$-foliated formations ($\Omega$-foliated Fitting classes) of groups which were constructed in [17] where $\Omega$ is a nonempty class of simple groups. O.V. Kamozina studied the lattice properties of $\omega$-fibered Fitting classes and $\Omega$-foliated Fitting classes (see, for example, [6; 7]). Yu.A. Elovikova studied the properties of the lattice of $\Omega$-foliated formations (see, for instance, [10; 11]).

The goal of this work is to study the lattice properties of $\omega$-fibered formations of finite groups. Let $\mathfrak{E}$ be the class of all finite groups, $\mathbb{P}$ is the set of all primes, $f : \omega \cup \{\omega'\} \to \{$formations of groups$\}$, where $f(\omega') \neq \varnothing$ (the symbol $\omega'$, following the terminology of [18], denotes an element from the domain $f$ that does not belong to $\omega$), $\delta : \mathbb{P} \to \{$nonempty Fitting formations$\}$ are functions called *$\omega F$-function* and *$\mathbb{P}FR$-function* respectively. A formation

$$\mathfrak{F} = \big(G \in \mathfrak{E} \mid G/O_\omega(G) \in f(\omega') \text{ and } G/G_{\delta(p)} \in f(p) \text{ for all } p \in \omega \cap \pi(G)\big)$$

is called an *$\omega$-fibered formation with a direction $\delta$* (briefly, an *$\omega\delta$-fibered formation*) *with an $\omega$-satellite* $f$ and denoted by $\mathfrak{F} = \omega F(f, \delta)$ [18], where $O_\omega(G)$ is the largest normal $\omega$-subgroup of the group $G$, $G_{\delta(p)}$ is the $\delta(p)$-radical of the group $G$, i.e. the largest normal subgroupe of the group $G$ belonging to the class $\delta(p)$, $\pi(G)$ is the set of all prime divisors of the order of the group $G$. Let $\mathfrak{F}$ be a nonempty class of groups and $\{\mathfrak{F}_i \mid i \in I\}$ be a set of nonempty subclasses of the class $\mathfrak{F}$. Put $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ if for any different $i, j \in I$ it is true that $\mathfrak{F}_i \cap \mathfrak{F}_j = (1)$ and every group $G \in \mathfrak{F}$ has a structure $G = A_{i_1} \times \ldots \times A_{i_t}$, where $A_{i_1} \in \mathfrak{F}_{i_1}, \ldots, A_{i_t} \in \mathfrak{F}_{i_t}$ for some $i_1, \ldots, i_t \in I$ [13].

In this work the following tasks have been solved. We have proved the modularity of the lattice $\Theta_{\omega\delta}$ of all $\omega$-fibered formations of finite groups with *bp*-direction $\delta$ such that $\delta_1 \leq \delta \leq \delta_3$ (Theorem 1). In the case $\delta = \delta_1$ Theorem 1 implies the result for $\omega$-local formations [15, Theorem 4, the case $n = 1$]. Note that according to [19, Theorem 3], there exists an infinite set of *bp*-directions $\delta$ satisfying the condition $\delta_1 \leq \delta \leq \delta_3$. In Theorems 2 and 3 we have established the sufficient conditions under which the lattice $\Theta_{\omega\delta}(\mathfrak{F})$ of all $\omega\delta$-fibered subformations of the $\omega$-fibered formation $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ with $bp$-direction $\delta$ is distributive and it is a lattice with complements respectively.

## 1. Notations and Definitions

Only finite groups are considered. The symbol := means the equality by the definition. Notations and definitions of groups and classes of groups are standard (see, for instance, [2]). We give just some of them. A group $G$ is called *monolithic* if $G$ has a unique minimal normal subgroup (*monolith*). Let $p \in \mathbb{P}$. By $Z_p$ we denote the group of the order $p$; $\mathfrak{N}_p$ is the class of all $p$-groups; $\mathfrak{E}_{p'}$ is the class of all $p'$-groups; $\mathfrak{E}_{(Z_p)'}$ is the class of all such groups that don't have composition factors which are isomorphic to $Z_p$; $\mathfrak{S}_{cp}$ is the class of all groups whose every chief $p$-factor is central. The class generated by the set $\mathfrak{X}$ of groups is denoted by $(\mathfrak{X})$, i.e. $(\mathfrak{X})$ is an intersection of all classes of groups containing $\mathfrak{X}$; in particular, $(G)$ is the class of all groups which are isomorphic to the group $G$; $(1)$ is the class of all identity groups. The formation generated by the set $\mathfrak{X}$ of groups is denoted by $form(\mathfrak{X})$, i.e. $form(\mathfrak{X})$ is an intersection of all formations containing $\mathfrak{X}$ [8]. For the class $\mathfrak{X}$ we put $\pi(\mathfrak{X}) := \cup_{G\in\mathfrak{X}}\pi(G)$. By $\mathfrak{X}_1\mathfrak{X}_2$ we denote the product of classes $\mathfrak{X}_1$ and $\mathfrak{X}_2$, i.e.

$$\mathfrak{X}_1\mathfrak{X}_2 = (\ G \in \mathfrak{E} \mid \text{there exists } N \triangleleft G \text{ such that } N \in \mathfrak{X}_1, G/N \in \mathfrak{X}_2\ ) \ [2].$$

Further, $\omega$ is a nonempty subset of the set $\mathbb{P}$. A group $G$ is called an *$\omega$-group* if $\pi(G) \subseteq \omega$; $\mathfrak{E}_\omega$ is the class of all $\omega$-groups. An $\omega$-satellite $f$ of the $\omega\delta$-fibered formation $\mathfrak{F}$ is called *inner* if $f(x) \subseteq \mathfrak{F}$ for any $x \in \omega \cup \{\omega'\}$. By $\omega F(\mathfrak{X}, \delta)$ it is denoted an $\omega\delta$-fibered formation generated by the set $\mathfrak{X}$ of groups, i.e. $\omega F(\mathfrak{X}, \delta)$ is an intersection of all $\omega\delta$-fibered formations containing $\mathfrak{X}$ [18]. Let $f_1$ and $f_2$ be $\omega F$-functions ($\mathbb{P}FR$-functions). We put $f_1 \le f_2$ if $f_1(x) \subseteq f_2(x)$ for every $x \in \omega\cup\{\omega'\}$ (for every $x \in \mathbb{P}$); we put $f_1 < f_2$ if $f_1 \le f_2$ and $f_1 \ne f_2$ [18]. An $\omega$-fibered formation with the direction $\delta$ is called: *$\omega$-absolute* if $\delta = \delta_0$ where $\delta_0(p) = \mathfrak{E}_{p'}$ for any $p \in \mathbb{P}$; *$\omega$-local* if $\delta = \delta_1$ where $\delta_1(p) = \mathfrak{E}_{p'}\mathfrak{N}_p$ for any $p \in \mathbb{P}$; *$\omega$-special* if $\delta = \delta_2$ where $\delta_2(p) = \mathfrak{E}_{(Z_p)'}\mathfrak{N}_p$ for any $p \in \mathbb{P}$; *$\omega$-central* if $\delta = \delta_3$ where $\delta_3(p) = \mathfrak{S}_{cp}$ for any $p \in \mathbb{P}$ [18]. It follows directly from these definitions that $\delta_0 < \delta_1 < \delta_2 < \delta_3$. The direction $\delta$ of the $\omega$-fibered formation is called a *bp-direction* if $\delta$ is a *b-direction*, i.e. $\delta(p)\mathfrak{N}_p = \delta(p)$ for any $p \in \mathbb{P}$, and $\delta$ is a *p-direction*, i.e. $\mathfrak{E}_{p'}\delta(p) = \delta(p)$ for any $p \in \mathbb{P}$ [19]. Further, we will use the following well-known examples of $\omega$-fibered formations.

**Example 1.** 1) Let $p \in \omega$. Then the class $\mathfrak{N}_p$ is an $\omega$-fibered formation with the $\omega$-satellite $f$ and the direction $\delta$ where $\delta$ is a $b$-direction and $f$ is an $\omega F$-function which has the following structure: $f(\omega') = \mathfrak{N}_p$, $f(p) = (1)$ and $f(q) = \varnothing$ for any $q \in \omega \setminus \{p\}$.

2) The class $\mathfrak{E}$ of all finite groups is an $\omega$-fibered formation with the $\omega$-satellite $f$ and the direction $\delta$ where $\delta$ is an arbitrary $\mathbb{P}FR$-function and $f$ is an $\omega F$-function which has the following structure: $f(p) = \mathfrak{E}$ for any $p \in \omega$, $f(\omega') = \mathfrak{E}$.

3) The class $(1)$ of all identity groups is an $\omega$-fibered formation with the $\omega$-satellite $f$ and the direction $\delta$ where $\delta$ is an arbitrary $\mathbb{P}FR$-function and $f$ is an $\omega F$-function which has the following structure: $f(p) = \varnothing$ for any $p \in \omega$, $f(\omega') = (1)$.

According to [1], in the lattice $\Theta$ the lattice intersection of the elements $x$ and $y$ is denoted by $x \wedge_\Theta y$ and the lattice union of the elements $x$ and $y$ is denoted by $x \vee_\Theta y$. A lattice $\Theta$ is called *distributive* if for any $x, y, z \in \Theta$ the following equality is true:

$$x \wedge_\Theta (y \vee_\Theta z) = (x \wedge_\Theta y) \vee_\Theta (x \wedge_\Theta z).$$

Note that, by [1, Theorem 9], the last equality is equivalent to the following equality:

$$x \vee_\Theta (y \wedge_\Theta z) = (x \vee_\Theta y) \wedge_\Theta (x \vee_\Theta z).$$

A lattice $\Theta$ is called *modular* if for any elements $x, y, z \in \Theta$ such that $y \le x$ it is true that

$$x \wedge_\Theta (y \vee_\Theta z) = y \vee_\Theta (x \wedge_\Theta z).$$

It follows directly from the definitions that any distributive lattice is a modular lattice. The smallest (the largest) element of the lattice is called a *zero* (*an identity*) of this lattice. Let $\Theta$ be a lattice with a zero $O$. An element $a \in \Theta$ is called an *atom* of the lattice $\Theta$ if $a \neq O$ and there is no such an element $x \in \Theta$ that $O < x < a$. A lattice $\Theta$ with a zero $O$ and an identity $I$ is called a *lattice with complements* if for any element $x \in \Theta$ there exists an element $y \in \Theta$ such that $x \wedge_\Theta y = O$ and $x \vee_\Theta y = I$; the element $y$ is called a *complement* for element $x$ in the lattice $\Theta$ [1].

Let $\Theta$ be a nonempty set of formations which is partially ordered regarding the inclusion $\subseteq$, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are $\Theta$-formations (i.e. $\mathfrak{F}_1, \mathfrak{F}_2 \in \Theta$). Then the lattice intersection and the lattice union of the formations $\mathfrak{F}_1$ и $\mathfrak{F}_2$ are defined respectively as

$$\mathfrak{F}_1 \wedge_\Theta \mathfrak{F}_2 := \mathfrak{F}_1 \cap \mathfrak{F}_2 \quad \text{(a)}, \qquad \mathfrak{F}_1 \vee_\Theta \mathfrak{F}_2 := \Theta form(\mathfrak{F}_1 \cup \mathfrak{F}_2) \quad \text{(b)},$$

where $\Theta form(\mathfrak{F}_1 \cup \mathfrak{F}_2)$ is a $\Theta$-formation generated by the union $\mathfrak{F}_1 \cup \mathfrak{F}_2$, i.e. $\Theta form(\mathfrak{F}_1 \cup \mathfrak{F}_2)$ is an intersection of all $\Theta$-formations containing $\mathfrak{F}_1 \cup \mathfrak{F}_2$. The set of formations $\Theta$ is called *a complete lattice of formations* if the intersection of any set of $\Theta$-formations is a $\Theta$-formation and there exists a formation $\mathfrak{M} \in \Theta$ such that $\mathfrak{F} \subseteq \mathfrak{M}$ for any formation $\mathfrak{F} \in \Theta$ [13]. For a given lattice $\Theta$ of formations by $\Theta(\mathfrak{F})$ we denote the set of all $\Theta$-subformations of the formation $\mathfrak{F}$.

In the following two lemmas the known results of the theory of formations of finite groups and some well-known properties of $\omega$-fibered formations are represented respectively.

**Lemma 1.** (1) *The lattice of all formations of finite groups is modular but is not distributive* [9, Corollary 9.9].

(2) *Suppose that* $\mathfrak{F} = \oplus_{i \in I} \mathfrak{F}_i$ *and* $\mathfrak{M}$ *is a nonempty subformation of the formation* $\mathfrak{F}$. *Then* $\mathfrak{M} = \oplus_{i \in I}(\mathfrak{F}_i \cap \mathfrak{M})$ [13, Lemma 4.3.4].

**Lemma 2.** (1) *Let* $\mathfrak{X}$ *be a nonempty class of group. Then the* $\omega$*-fibered formation* $\mathfrak{F} = \omega F(\mathfrak{X}, \delta)$ *with the direction* $\delta$*, where* $\delta_0 \leq \delta$*, has a unique minimal* $\omega$*-satellite* $f$ *such that*

$$f(\omega') = form(G/O_\omega(G) \mid G \in \mathfrak{X}), f(p) = form(G/G_{\delta(p)} \mid G \in \mathfrak{X})$$

*for all* $p \in \omega \cap \pi(\mathfrak{X})$ *and* $f(p) = \varnothing$ *if* $p \in \omega \setminus \pi(\mathfrak{X})$ [18, Theorem 5].

(2) *Let* $f_i$ *be a minimal* $\omega$*-satellite of the* $\omega$*-fibered formation* $\mathfrak{F}_i$ *with the direction* $\delta$ *where* $\delta_0 \leq \delta$*,* $i = 1, 2$. *Then* $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ *if and only if* $f_1 \leq f_2$ [18, Corollary 5.1].

(3) *Let* $\mathfrak{F} = \omega F(f, \delta)$ *be an* $\omega$*-fibered formation with the inner* $\omega$*-satellite* $f$ *and with the bp-direction* $\delta$ *such that* $\delta_1 \leq \delta \leq \delta_3$. *Then* $\mathfrak{F}$ *has a unique maximal inner* $\omega$*-satellite* $h$ *such that* $h(\omega') = \mathfrak{F}$*,* $h(p) = \mathfrak{N}_p h(p) = \mathfrak{N}_p f(p)$ *for any* $p \in \omega$ [19, Theorem 6].

(4) *Let* $\mathfrak{F} = \omega F(f, \delta)$ *where* $\delta$ *is an arbitrary* $\mathbb{P}FR$*-function. Then* $\mathfrak{F} = \omega F(h, \delta)$ *where* $h(\omega') = \mathfrak{F}$ *and* $h(p) = f(p)$ *for any* $p \in \omega$ [18, Lemma 4 (2)].

(5) *Let* $\delta$ *be an arbitrary* $\mathbb{P}FR$*-function,* $\mathfrak{F} = \cap_{i \in I} \mathfrak{F}_i$ *where* $\mathfrak{F}_i = \omega F(f_i, \delta)$*,* $i \in I$. *Then* $\mathfrak{F} = \omega F(f, \delta)$ *where* $f = \cap_{i \in I} f_i$ [18, Lemma 5].

(6) *Let* $\mathfrak{F}$ *be an* $\omega$*-fibered formation with an inner* $\omega$*-satellite* $f$ *and with a bp-direction* $\delta$. *Then* $\mathfrak{N}_p f(p) \subseteq \mathfrak{F}$ *for all* $p \in \omega$ [19, Lemma 6 (1)].

Let $\delta$ be an arbitrary $\mathbb{P}FR$-function. Denote by $\Theta_{\omega\delta}$ the set of all $\omega\delta$-fibered formations. Since the formations (1) and $\mathfrak{E}$ belong to $\Theta_{\omega\delta}$ (see Example 1) then the set $\Theta_{\omega\delta}$ is nonempty. If $\mathfrak{F}$ is a nonempty formation then $(1) \in \Theta_{\omega\delta}(\mathfrak{F})$ and, therefore, $\Theta_{\omega\delta}(\mathfrak{F}) \neq \varnothing$. According to Lemma 2 (5), the intersection of any set of $\omega\delta$-fibered formations is an $\omega\delta$-fibered formation. Then, in view of (a) and (b), we conclude that the set $\Theta_{\omega\delta}$ is a lattice with the zero (1) and with the identity $\mathfrak{E}$ and, consequently, is a complete lattice of formations. If $\mathfrak{F}$ is a nonidentity $\omega\delta$-formation then $\Theta_{\omega\delta}(\mathfrak{F})$ is a lattice (a complete lattice) with the zero (1) and with the identity $\mathfrak{F}$. Denote the set of all formations of finite groups by $\Theta_{\mathfrak{E}}$. Note that $\Theta_{\mathfrak{E}}$ is a complete lattice of formations with the zero $\varnothing$ and with the identity $\mathfrak{E}$.

Let $\Theta$ be a complete lattice of formations. An $\omega F$-function $f$ is called $\Theta$-*valuable* if all its nonempty values belong to $\Theta$. Let $\{f_i \mid i \in I\}$ is a set of $\Theta$-valuable $\omega F$-functions. In accordance with the notation adopted in [13], by $\vee_{\Theta_{i\in I}} f_i$ we denote such an $\omega F$-function $f$ that for any $x \in \omega \cup \{\omega'\}$ it is true that $f(x) = \vee_{\Theta_{i\in I}} f_i(x)$, i.e. $f(x) = \Theta form(\cup_{i\in I} f_i(x))$ for any $x \in \omega \cup \{\omega'\}$. Following [13], a complete lattice $\Theta$ of formations is called $\omega\delta$-*inductive* if for any set $\{\mathfrak{F}_i \mid i \in I\}$ of $\omega\delta$-fibered formations which have at least one $\Theta$-valuable $\omega$-satellite and for any set $\{f_i \mid i \in I\}$ where $f_i$ is an inner $\Theta$-valuable $\omega$-satellite of the formation $\mathfrak{F}_i$, $i \in I$, the following equality takes place: $\vee_{\Theta_{\omega\delta}}{}_{i\in I}(\mathfrak{F}_i) = \omega F(\vee_{\Theta_{i\in I}} f_i, \delta)$, i.e.

$$\omega F(\cup_{i\in I}\mathfrak{F}_i, \delta) = \omega F(f, \delta), \text{ where } f := \vee_{\Theta_{i\in I}} f_i.$$

## 2. Modularity of the lattice $\Theta_{\omega\delta}$

First we will prove the following two lemmas.

**Lemma 3.** *Let $\delta$ be a bp-direction such that $\delta_1 \le \delta \le \delta_3$. Then the set $\Theta_{\mathfrak{E}}$ of all formations of groups is an $\omega\delta$-inductive lattice.*

**Proof.** As mentioned above, the set $\Theta_{\mathfrak{E}}$ is a complete lattice of formations. Put $\mathfrak{X} := \vee_{\Theta_{\omega\delta}}{}_{i\in I}(\mathfrak{F}_i)$ where $\mathfrak{F}_i := \omega F(f_i, \delta)$ and $f_i$ is an inner $\omega$-satellite of the formation $\mathfrak{F}_i$, $i \in I$. Note that, by the definition of an $\omega F$-function, $f_i$ is a $\Theta_{\mathfrak{E}}$-valuable $\omega$-satellite of the formation $\mathfrak{F}_i$, $i \in I$. Put $\mathfrak{Y} := \omega F(f, \delta)$ where $f := \vee_{\Theta_{\mathfrak{E}}}{}_{(i\in I)} f_i$. Then

$$f(x) = \vee_{\Theta_{\mathfrak{E}}}{}_{(i\in I)} f_i(x) = form(\cup_{i\in I} f_i(x)) \text{ for any } x \in \omega \cup \{\omega'\}.$$

Prove that $\mathfrak{X} = \mathfrak{Y}$. Since $\mathfrak{X} = \vee_{\Theta_{\omega\delta}}{}_{i\in I}(\mathfrak{F}_i)$ then $\mathfrak{X} = \omega F(\cup_{i\in I}\mathfrak{F}_i, \delta)$. In a view of $f_i \le f$, we have $\mathfrak{F}_i \subseteq \mathfrak{Y}$, $i \in I$, and so $\cup_{i\in I}\mathfrak{F}_i \subseteq \mathfrak{Y}$. Since $\mathfrak{X}$ is the smallest $\omega\delta$-fibered formation which contains the union $\cup_{i\in I}\mathfrak{F}_i$ then $\mathfrak{X} \subseteq \mathfrak{Y}$.

Establish that $\mathfrak{Y} \subseteq \mathfrak{X}$. Since $\delta_0 \le \delta_1 \le \delta$ then, according to Lemma 2 (1), there exists a unique minimal $\omega$-satellite $h$ of the formation $\mathfrak{X}$ and there exists a unique minimal $\omega$-satellite $h_i$ of the formation $\mathfrak{F}_i$, $i \in I$. According to Lemma 2 (3) the formation $\mathfrak{X}$ has a unique maximal inner $\omega$-satellite $m$, moreover, $m(p) = \mathfrak{N}_p h(p)$ for any $p \in \omega$ and $m(\omega') = \mathfrak{X}$; the formation $\mathfrak{F}_i$ has a unique maximal inner $\omega$-satellite $m_i$ and $m_i(p) = \mathfrak{N}_p h_i(p)$ for any $p \in \omega$, $i \in I$. Verify that $f \le m$. It is sufficient to show that $f(x) \subseteq m(x)$ for any $x \in \omega \cup \{\omega'\}$.

Put $p \in \omega$. Consider the case $p \in \omega \setminus \pi(\cup_{i\in I}\mathfrak{F}_i)$. By Lemma 2 (1), $h(p) = \varnothing$, and so $m(p) = \mathfrak{N}_p h(p) = \varnothing$. Since $p \in \omega \setminus \pi(\cup_{i\in I}\mathfrak{F}_i)$ then $p \in \omega\setminus\pi(\mathfrak{F}_i)$ for any $i \in I$. According to Lemma 2 (1), it meens that $h_i(p) = \varnothing$, and so $m_i(p) = \mathfrak{N}_p h_i(p) = \varnothing$ for any $i \in I$. Since $f_i$ is the inner $\omega$-satellite of $\mathfrak{F}_i$ and $m_i$ is the unique maximal inner $\omega$-satellite of $\mathfrak{F}_i$ then $f_i \le m_i$ and, therefore, $f_i(p) = \varnothing$ for any $i \in I$. This implies that $f(p) = \varnothing$. Thus, in the case $p \in \omega \setminus \pi(\cup_{i\in I}\mathfrak{F}_i)$ we obtained $f(p) = \varnothing = m(p)$.

Suppose that $p \in \omega \cap \pi(\cup_{i\in I}\mathfrak{F}_i)$ and $J = \{j \in I \mid p \in \pi(\mathfrak{F}_j)\}$. Then, according to Lemma 2 (1), $h_j(p) \ne \varnothing$ for any $j \in J$, $h_k(p) = \varnothing$ for any $k \in I \setminus J$, and it is true that

$$form(\ \cup_{i\in I} h_i(p)\ ) = form(\ \cup_{j\in J}\ (form(G/G_{\delta(p)}) \mid G \in \mathfrak{F}_j)\ )$$

$$\subseteq form(\ G/G_{\delta(p)} \mid G \in \cup_{i\in I}\mathfrak{F}_i\ ) = h(p).$$

Since $f_i \le m_i$, $i \in I$, then

$$f(p) = form(\ \cup_{i\in I} f_i(p)\ ) \subseteq form(\ \cup_{i\in I} m_i(p)\ )$$

$$= form(\ \cup_{i\in I}\ \mathfrak{N}_p h_i(p)\ ) \subseteq \mathfrak{N}_p form(\ \cup_{i\in I} h_i(p)\ ) \subseteq \mathfrak{N}_p h(p) = m(p).$$

Thus, $f(p) \subseteq m(p)$ for any $p \in \omega$. Since $f_i$ is an inner $\omega$-satellite of the formation $\mathfrak{F}_i$ then $f_i(\omega') \subseteq \mathfrak{F}_i \subseteq \mathfrak{X}$, $i \in I$. Hence, in view of $\mathfrak{X} \in \Theta_{\mathfrak{E}}$, we conclude that $f(\omega') = form(\cup_{i\in I} f_i(\omega')) \subseteq \mathfrak{X} = m(\omega')$. Thus, $f(x) \subseteq m(x)$ for every $x \in \omega \cup \{\omega'\}$. Therefore, $f \leq m$. It means that $\mathfrak{Y} \subseteq \mathfrak{X}$. So, it has been established that $\mathfrak{X} = \mathfrak{Y}$. Consequently, the lattice $\Theta_{\mathfrak{E}}$ of all formations of groups is $\omega\delta$-inductive.

The lemma is proved.

**Theorem 1.** *Let $\delta$ be a bp-direction such that $\delta_1 \leq \delta \leq \delta_3$. Then the set $\Theta_{\omega\delta}$ of all $\omega\delta$-fibered formations of groups is a modular lattice.*

**Proof.** As mentioned above, the set $\Theta_{\omega\delta}$ is a lattice. Let $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \in \Theta_{\omega\delta}$, and $\mathfrak{F}_2 \subseteq \mathfrak{F}_1$. Prove that

$$\mathfrak{F}_1 \wedge_{\Theta_{\omega\delta}} (\mathfrak{F}_2 \vee_{\Theta_{\omega\delta}} \mathfrak{F}_3) = \mathfrak{F}_2 \vee_{\Theta_{\omega\delta}} (\mathfrak{F}_1 \wedge_{\Theta_{\omega\delta}} \mathfrak{F}_3).$$

Put $\mathfrak{H} := \mathfrak{F}_2 \vee_{\Theta_{\omega\delta}} \mathfrak{F}_3$, $\mathfrak{M} := \mathfrak{F}_1 \wedge_{\Theta_{\omega\delta}} \mathfrak{F}_3$, $\mathfrak{X} := \mathfrak{F}_1 \wedge_{\Theta_{\omega\delta}} \mathfrak{H}$, $\mathfrak{Y} := \mathfrak{F}_2 \vee_{\Theta_{\omega\delta}} \mathfrak{M}$. Verify that $\mathfrak{X} = \mathfrak{Y}$. Let $f_i$ be a minimal $\omega$-satellite of the formation $\mathfrak{F}_i$, $i = 1, 2, 3$, and put

$$h := f_2 \vee_{\Theta_{\mathfrak{E}}} f_3, \quad m := f_1 \wedge_{\Theta_{\mathfrak{E}}} f_3, \quad x := f_1 \wedge_{\Theta_{\mathfrak{E}}} h, \quad y := f_2 \vee_{\Theta_{\mathfrak{E}}} m.$$

First we will establish that $\mathfrak{X} = \omega F(x, \delta)$ and $\mathfrak{Y} = \omega F(y, \delta)$. Indeed, since by Lemma 3 the lattice $\Theta_{\mathfrak{E}}$ is $\omega\delta$-inductive then $\mathfrak{H} = \omega F(h, \delta)$ and, therefore, according to Lemma 2 (5), we obtain $\mathfrak{X} = \omega F(x, \delta)$. By Lemma 2 (5) it is true that $\mathfrak{M} = \omega F(m, \delta)$ and, in view of Lemma 3, we conclude that $\mathfrak{Y} = \omega F(y, \delta)$. Verify that $x = y$. Put $p \in \omega \cup \{\omega'\}$. By inclusion $\mathfrak{F}_2 \subseteq \mathfrak{F}_1$ and according to Lemma 2 (2), we have that $f_2 \leq f_1$ and, consequently, $f_2(p) \subseteq f_1(p)$. By Lemma 1 (1) the lattice $\Theta_{\mathfrak{E}}$ is modular. Then

$$x(p) = (f_1 \wedge_{\Theta_{\mathfrak{E}}} (f_2 \vee_{\Theta_{\mathfrak{E}}} f_3))(p) = (f_2 \vee_{\Theta_{\mathfrak{E}}} (f_1 \wedge_{\Theta_{\mathfrak{E}}} f_3))(p) = y(p).$$

Thus, $x = y$ and, therefore, $\mathfrak{X} = \omega F(x, \delta) = \omega F(y, \delta) = \mathfrak{Y}$. Hence, the lattice $\Theta_{\omega\delta}$ is modular.

The theorem is proved.

## 3. Distributivity of the lattice $\Theta_{\omega\delta}(\mathfrak{F})$

For an $\omega$-fibered formation $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ with a *bp*-direction $\delta$ we will establish sufficient conditions under which the lattice $\Theta_{\omega\delta}(\mathfrak{F})$ of all $\omega\delta$-fibered subformations of the formation $\mathfrak{F}$ is distributive. We will prove the following two lemmas first.

**Lemma 4.** *Let $\delta$ be a $\mathbb{P}FR$-function such that $\delta_0 \leq \delta$, and suppose that $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ where $\mathfrak{F}_i$ is an atom of the lattice $\Theta_{\omega\delta}(\mathfrak{F})$, $i \in I$. If $(1) \neq \mathfrak{H} \in \Theta_{\omega\delta}(\mathfrak{F})$ then there exists such a set $J \subseteq I$ that $\mathfrak{H} = \oplus_{j\in J}\mathfrak{F}_j$.*

**Proof.** Let $\mathfrak{H} \in \Theta_{\omega\delta}(\mathfrak{F})$ and $\mathfrak{H} \neq (1)$. According to Lemma 1 (2), we obtain $\mathfrak{H} = \oplus_{i\in I}(\mathfrak{F}_i \cap \mathfrak{H})$. Since $\mathfrak{F}_i \cap \mathfrak{H} \subseteq \mathfrak{F}_i$ and by Lemma 2 (5) $\mathfrak{F}_i \cap \mathfrak{H} \in \Theta_{\omega\delta}$ then, in view of the fact that $\mathfrak{F}_i$ is an atom of the lattice $\Theta_{\omega\delta}(\mathfrak{F})$, it is true that $\mathfrak{F}_i \cap \mathfrak{H} = (1)$ or $\mathfrak{F}_i \cap \mathfrak{H} = \mathfrak{F}_i$. Put $J = \{j \in I \mid \mathfrak{F}_j \cap \mathfrak{H} = \mathfrak{F}_j\}$. Then $\mathfrak{H} = \oplus_{j\in J}(\mathfrak{F}_j \cap \mathfrak{H}) = \oplus_{j\in J}\mathfrak{F}_j$.

The lemma is proved.

**Lemma 5.** *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be such $\omega$-fibered formations with a bp-direction $\delta$ that $\mathfrak{F}_1 \cap \mathfrak{F}_2 = (1)$. Then $\pi(\mathfrak{F}_1) \cap \pi(\mathfrak{F}_2) \cap \omega = \varnothing$.*

**Proof.** Let $f_i$ be a minimal $\omega$-satellite of the formation $\mathfrak{F}_i$, $i = 1, 2$. Suppose that $\pi(\mathfrak{F}_1) \cap \pi(\mathfrak{F}_2) \cap \omega \neq \varnothing$. Then there exists such a prime number $p$ that $p \in \pi(\mathfrak{F}_1) \cap \pi(\mathfrak{F}_2) \cap \omega$. Since $\pi(\mathfrak{F}_1) \cap \pi(\mathfrak{F}_2) \cap \omega \subseteq \pi(\mathfrak{F}_i) \cap \omega$ then $p \in \pi(\mathfrak{F}_i) \cap \omega$ and, in view of Lemma 2 (1), we obtain $f_i(p) \neq \varnothing$, $i = 1, 2$. Then, according to Lemma 2 (6), we conclude that $\mathfrak{N}_p \subseteq \mathfrak{N}_p f_i(p) \subseteq \mathfrak{F}_i$, $i = 1, 2$. Therefore, $\mathfrak{N}_p \subseteq \mathfrak{F}_1 \cap \mathfrak{F}_2 = (1)$. Contradiction. Consequently, $\pi(\mathfrak{F}_1) \cap \pi(\mathfrak{F}_2) \cap \omega = \varnothing$.

The lemma is proved.

**Theorem 2.** *Let $\delta$ be a bp-direction, $(1) \neq \mathfrak{F} \in \Theta_{\omega\delta}$, and suppose that $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ where $\{\mathfrak{F}_i \mid i \in I\}$ is the set of all atoms of the lattice $\Theta_{\omega\delta}(\mathfrak{F})$. If $\pi(\mathfrak{F}_i) \cap \omega \neq \varnothing$ for any $i \in I$ then $\Theta_{\omega\delta}(\mathfrak{F})$ is a distributive lattice.*

**Proof.** Put $\Theta := \Theta_{\omega\delta}(\mathfrak{F})$. As mentioned above, the set $\Theta$ is a lattice. Let $\mathfrak{M}, \mathfrak{H}, \mathfrak{X} \in \Theta$. Verify that

$$(\mathfrak{M} \wedge_\Theta \mathfrak{X}) \vee_\Theta (\mathfrak{M} \wedge_\Theta \mathfrak{H}) = \mathfrak{M} \wedge_\Theta (\mathfrak{X} \vee_\Theta \mathfrak{H}) \qquad \text{(I)}.$$

If at least one of the formations $\mathfrak{M}$, $\mathfrak{H}$ or $\mathfrak{X}$ is an identity formation then equality (I) is true. Let $\mathfrak{M}$, $\mathfrak{H}$, $\mathfrak{X}$ be nonidentity formations. Let us introduce the following notations:

$$\mathscr{X} := (\mathfrak{M} \wedge_\Theta \mathfrak{X}) \vee_\Theta (\mathfrak{M} \wedge_\Theta \mathfrak{H}), \quad \mathscr{Y} := \mathfrak{M} \wedge_\Theta (\mathfrak{X} \vee_\Theta \mathfrak{H}),$$

i.e. $\mathscr{X}$ is a $\Theta$-formation generated by the set $(\mathfrak{M} \cap \mathfrak{X}) \cup (\mathfrak{M} \cap \mathfrak{H})$, $\mathscr{Y}$ is an intersection of the formations $\mathfrak{M}$ and $\mathfrak{X} \vee_\Theta \mathfrak{H}$ where $\mathfrak{X} \vee_\Theta \mathfrak{H} := \mathscr{Z}$ is a $\Theta$-formation generated by the set $\mathfrak{X} \cup \mathfrak{H}$.

Prove that $\mathscr{X} \subseteq \mathscr{Y}$. In view of the inclusions $\mathfrak{M} \cap \mathfrak{X} \subseteq \mathfrak{M}$ and $\mathfrak{M} \cap \mathfrak{H} \subseteq \mathfrak{M}$ it is true that $(\mathfrak{M}\cap\mathfrak{X}) \cup (\mathfrak{M}\cap\mathfrak{H}) \subseteq \mathfrak{M}$. Since $\mathfrak{M} \in \Theta$ we conclude that $\mathscr{X} \subseteq \mathfrak{M}$. Further, since $\mathfrak{M}\cap\mathfrak{X} \subseteq \mathfrak{X} \subseteq \mathfrak{X}\cup\mathfrak{H}$ and $\mathfrak{M} \cap \mathfrak{H} \subseteq \mathfrak{H} \subseteq \mathfrak{X} \cup \mathfrak{H}$ then $(\mathfrak{M} \cap \mathfrak{X}) \cup (\mathfrak{M} \cap \mathfrak{H}) \subseteq \mathfrak{X} \cup \mathfrak{H} \subseteq \mathscr{Z}$ and $\mathscr{X} \subseteq \mathscr{Z}$. Consequently, $\mathscr{X} \subseteq \mathfrak{M} \cap \mathscr{Z} = \mathscr{Y}$.

Suppose that $\mathscr{X} \subset \mathscr{Y}$ and $G$ is a group of the minimal order in $\mathscr{Y} \setminus \mathscr{X}$. Then $G \neq 1$. Since $\mathscr{X}$ and $\mathscr{Y}$ are formations then $G$ is a monolithic group. Since $G \in \mathscr{Y} = \mathfrak{M} \cap \mathscr{Z}$ then $G \in \mathfrak{M}$ and $G \in \mathscr{Z} = \Theta form(\mathfrak{X} \cup \mathfrak{H})$. From $G \in \mathfrak{M} \subseteq \mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ and $G$ is a monolithic group, it follows that $G \in \mathfrak{F}_j$ for some $j \in I$. Consider the formation $\Theta form(G)$. Since $(1) \subset \Theta form(G) \subseteq \mathfrak{F}_j$ and $\mathfrak{F}_j$ is an atom of the lattice $\Theta$ then $\mathfrak{F}_j = \Theta form(G)$. Since $G \in \mathscr{Z}$ we conclude that $\mathfrak{F}_j \subseteq \mathscr{Z}$.

Put $\mathscr{Z}_1 := \omega F(\mathfrak{X} \cup \mathfrak{H}, \delta)$. Prove that $\mathscr{Z} = \mathscr{Z}_1$. Recall that

$$\mathscr{Z} = \Theta form(\mathfrak{X} \cup \mathfrak{H}) \text{ where } \Theta = \Theta_{\omega\delta}(\mathfrak{F}).$$

Since $\mathfrak{X} \cup \mathfrak{H} \subseteq \mathscr{Z}$ and $\mathscr{Z} \in \Theta_{\omega\delta}$ then $\mathscr{Z}_1 \subseteq \mathscr{Z}$. Verify that $\mathscr{Z} \subseteq \mathscr{Z}_1$. In view of $\mathfrak{X} \cup \mathfrak{H} \subseteq \mathscr{Z}_1$ it is sufficient to show that $\mathscr{Z}_1 \in \Theta$. Since $\mathscr{Z}_1 \in \Theta_{\omega\delta}$ it is sufficient to verify that $\mathscr{Z}_1 \subseteq \mathfrak{F}$. Indeed, $\mathfrak{F} \in \Theta_{\omega\delta}$. Since $\mathfrak{X} \subseteq \mathfrak{F}$ and $\mathfrak{H} \subseteq \mathfrak{F}$ we conclude that $\mathfrak{X} \cup \mathfrak{H} \subseteq \mathfrak{F}$. Consequently, $\mathscr{Z}_1 \subseteq \mathfrak{F}$, and so $\mathscr{Z}_1 \in \Theta$. It means that $\mathscr{Z}_1$ is a $\Theta$-formation which contains the set $\mathfrak{X} \cup \mathfrak{H}$. Hence, $\mathscr{Z} \subseteq \mathscr{Z}_1$. Thus, it has been established that $\mathscr{Z} = \mathscr{Z}_1$ and it is true that $\pi(\mathscr{Z}) = \pi(\mathscr{Z}_1)$. In view of Lemma 2 (1), $\pi(\mathscr{Z}_1) \cap \omega = \pi(\mathfrak{X} \cup \mathfrak{H}) \cap \omega$ and, therefore, $\pi(\mathscr{Z}) \cap \omega = \pi(\mathfrak{X} \cup \mathfrak{H}) \cap \omega$.

According to Lemma 4, there exist such subsets $T, R, S$ of the set $I$ that $\mathfrak{X} = \oplus_{t\in T}\mathfrak{F}_t$, $\mathfrak{H} = \oplus_{r\in R}\mathfrak{F}_r$, $\mathscr{Z} = \oplus_{s\in S}\mathfrak{F}_s$. Verify that $S = R \cup T$. Since $\mathfrak{X} \cup \mathfrak{H} \subseteq \mathscr{Z}$ then $R \cup T \subseteq S$. Suppose that $R \cup T \subset S$ and $s_1 \in S \setminus (R \cup T)$. Then $\mathfrak{F}_{s_1} \subseteq \mathscr{Z}$ but $\mathfrak{F}_{s_1} \nsubseteq \mathfrak{X}$ and $\mathfrak{F}_{s_1} \nsubseteq \mathfrak{H}$. Hence,

$$\pi(\mathfrak{F}_{s_1}) \cap \omega \;\subseteq\; \pi(\mathscr{Z}) \cap \omega \;=\; \pi(\mathfrak{X} \cup \mathfrak{H}) \cap \omega \;=\; (\pi(\mathfrak{X}) \cup \pi(\mathfrak{H})) \cap \omega \;=\; (\pi(\mathfrak{X}) \cap \omega) \cup (\pi(\mathfrak{H}) \cap \omega).$$

Therefore, $\pi(\mathfrak{F}_{s_1}) \cap \omega \subseteq \pi(\mathfrak{X}) \cap \omega$ or $\pi(\mathfrak{F}_{s_1}) \cap \omega \subseteq \pi(\mathfrak{H}) \cap \omega$. Consequently,

$$\pi(\mathfrak{F}_{s_1}) \cap \pi(\mathfrak{X}) \cap \omega = \pi(\mathfrak{F}_{s_1}) \cap \omega \neq \varnothing \text{ or } \pi(\mathfrak{F}_{s_1}) \cap \pi(\mathfrak{H}) \cap \omega = \pi(\mathfrak{F}_{s_1}) \cap \omega \neq \varnothing.$$

On the other hand, since $\mathfrak{F}_{s_1} \nsubseteq \mathfrak{X}$ and $\mathfrak{X} = \oplus_{t\in T}\mathfrak{F}_t$ we conclude that $\mathfrak{F}_{s_1} \cap \mathfrak{X} = (1)$. According to Lemma 5, $\pi(\mathfrak{F}_{s_1}) \cap \pi(\mathfrak{X}) \cap \omega = \varnothing$. Likewise, $\mathfrak{F}_{s_1} \cap \mathfrak{H} = (1)$ and, in view of Lemma 5, it is true that $\pi(\mathfrak{F}_{s_1}) \cap \pi(\mathfrak{H}) \cap \omega = \varnothing$. Contradiction. Thus, $S = R \cup T$.

Since $\mathfrak{F}_j \subseteq \mathscr{Z}$ then $j \in S$, and so $j \in R$ or $j \in T$. Thus, $\mathfrak{F}_j \subseteq \mathfrak{X}$ or $\mathfrak{F}_j \subseteq \mathfrak{H}$. Therefore, $G \in \mathfrak{X}$ or $G \in \mathfrak{H}$. Suppose that $G \in \mathfrak{X}$. Since $G \in \mathfrak{M}$ then $G \in \mathfrak{M} \cap \mathfrak{X} = \mathfrak{M} \wedge_\Theta \mathfrak{X} \subseteq \mathscr{X}$. Contradiction. If $G \in \mathfrak{H}$ then $G \in \mathfrak{M} \cap \mathfrak{H} = \mathfrak{M} \wedge_\Theta \mathfrak{H} \subseteq \mathscr{X}$. Contradiction. Thus, $\mathscr{X} = \mathscr{Y}$ and the equality (I) is true.

The theorem is proved.

## 4. The conditions under which the lattice $\Theta_{\omega\delta}(\mathfrak{F})$ is a lattice with complements

First we wil prove the following lemma.

**Lemma 6.** *Let $\mathfrak{F} = \mathfrak{H} \oplus \mathfrak{L}$ where $\mathfrak{H}, \mathfrak{L}$ are nonidentity formations, $\pi(\mathfrak{H}) \cap \omega \neq \varnothing$, $\pi(\mathfrak{L}) \cap \omega \neq \varnothing$ and $\pi(\mathfrak{H}) \cap \pi(\mathfrak{L}) \cap \omega = \varnothing$. If $\mathfrak{F}, \mathfrak{H} \in \Theta_{\omega\delta}$ and $\delta$ is a bp-direction then $\mathfrak{L} \in \Theta_{\omega\delta}$.*

**Proof.** Let $\mathfrak{F}$, $\mathfrak{H} \in \Theta_{\omega\delta}$ and $\delta$ be a *bp*-direction. Prove that $\mathfrak{L} \in \Theta_{\omega\delta}$. Put $\mathfrak{F} := \omega F(f, \delta)$, $\mathfrak{H} := \omega F(h, \delta)$. According to Lemma 2 (4), we can put $f(\omega') = \mathfrak{F}$ and $h(\omega') = \mathfrak{H}$. Consider such an $\omega F$-function $m$ that $m(\omega') = \mathfrak{L}$, $m(q) = f(q)$ for any $q \in (\pi(\mathfrak{F}) \cap \omega) \setminus (\pi(\mathfrak{H}) \cap \omega)$ and $m(q) = \varnothing$ for any $q \in (\pi(\mathfrak{H}) \cap \omega) \cup (\omega \setminus (\pi(\mathfrak{F}))$. Put $\mathfrak{M} := \omega F(m, \delta)$.

Verify that $\mathfrak{M} \subseteq \mathfrak{F}$. Suppose that $M \in \mathfrak{M}$. Then

$$M/O_\omega(M) \in m(\omega') = \mathfrak{L} \subseteq \mathfrak{F} = f(\omega')$$

and $M/M_{\delta(p)} \in m(p)$ for any $p \in \pi(M) \cap \omega$. Hence, $m(p) \neq \varnothing$ for any $p \in \pi(M) \cap \omega$. Consequently, $m(p) = f(p)$ and it is true that $M/M_{\delta(p)} \in f(p)$ for any $p \in \pi(M) \cap \omega$. Thus, $M \in \mathfrak{F}$ and $\mathfrak{M} \subseteq \mathfrak{F}$.

Prove that $\mathfrak{L} = \mathfrak{M}$. Assume that $L \in \mathfrak{L}$. Since $\mathfrak{L}$ is a $Q$-closed class, we obtain $L/O_\omega(L) \in \mathfrak{L} = m(\omega')$. Put $p \in \pi(L) \cap \omega$. Then $p \in \pi(\mathfrak{L}) \cap \omega$. From $\pi(\mathfrak{H}) \cap \pi(\mathfrak{L}) \cap \omega = \varnothing$ it follows that $p \in (\pi(\mathfrak{F}) \cap \omega) \setminus (\pi(\mathfrak{H}) \cap \omega)$. Since $L \in \mathfrak{L} \subseteq \mathfrak{F}$ then we conclude that $L/L_{\delta(p)} \in f(p) = m(p)$. Thus, $L \in \mathfrak{M}$, and so $\mathfrak{L} \subseteq \mathfrak{M}$.

Suppose that $\mathfrak{L} \subset \mathfrak{M}$ and $K$ is a group of the minimal order in $\mathfrak{M} \setminus \mathfrak{L}$. Hence, $K \neq 1$ and $K$ is a monolithic group. Assume that $K$ is an $\omega'$-group. Then $O_\omega(K) = 1$. Since $K \in \mathfrak{M}$ then

$$K \cong K/O_\omega(K) \in m(\omega') = \mathfrak{L}$$

that is impossible. Consequently, $\pi(K) \cap \omega \neq \varnothing$. From $K \in \mathfrak{M}$ and $\mathfrak{M} \subseteq \mathfrak{F}$ it follows that $K \in \mathfrak{F}$. It means that $K = A \times B$ where $A \in \mathfrak{H}$, $B \in \mathfrak{L}$. Since $K$ is a monolithic group then $K = A$ or $K = B$. If $K = B$ then $K \in \mathfrak{L}$. Contradiction. Thus, $K = A \in \mathfrak{H}$. Then $\pi(K) \subseteq \pi(\mathfrak{H})$ and, therefore, for any $p \in \pi(K) \cap \omega \subseteq \pi(\mathfrak{H}) \cap \omega$ we obtain $m(p) = \varnothing$. On the other hand, from $K \in \mathfrak{M}$ it follows that $K/K_{\delta(p)} \in m(p)$ for any $p \in \pi(K) \cap \omega$. Contradiction. Thus, $\mathfrak{L} = \mathfrak{M}$.

The lemma is proved.

**Theorem 3.** *Let $\delta$ be a bp-direction, $(1) \neq \mathfrak{F} \in \Theta_{\omega\delta}$, $\mathfrak{F} = \oplus_{i \in I} \mathfrak{F}_i$ where $\{\mathfrak{F}_i \mid i \in I\}$ is the set of all atoms of the lattice $\Theta_{\omega\delta}(\mathfrak{F})$. If $\pi(\mathfrak{F}_i) \cap \omega \neq \varnothing$ for any $i \in I$ then $\Theta_{\omega\delta}(\mathfrak{F})$ is a lattice with complements.*

**Proof.** Put $\Theta := \Theta_{\omega\delta}(\mathfrak{F})$. Note that the lattice $\Theta$ has a zero (1) and an identity $\mathfrak{F}$. Let $\mathfrak{H} \in \Theta$. Prove that $\mathfrak{H}$ has a complement in the lattice $\Theta$. If $\mathfrak{H} = (1)$ then $\mathfrak{F}$ is a complement to $\mathfrak{H}$ in $\Theta$. If $\mathfrak{H} = \mathfrak{F}$ then (1) is a complement to $\mathfrak{H}$ in $\Theta$. Put $\mathfrak{H} \neq (1)$ and $\mathfrak{H} \neq \mathfrak{F}$. In view of Lemma 4, we obtain $\mathfrak{H} = \oplus_{i \in J} \mathfrak{F}_i$ where $J \subset I$, $J \neq \varnothing$. Put $I_1 := I \setminus J$ and $\mathfrak{M} := \oplus_{i \in I_1} \mathfrak{F}_i$. Then $\mathfrak{F} = (\oplus_{i \in J} \mathfrak{F}_i) \oplus (\oplus_{i \in I_1} \mathfrak{F}_i) = \mathfrak{H} \oplus \mathfrak{M}$. Verify that $\mathfrak{M}$ is a complement to $\mathfrak{H}$ in the lattice $\Theta$. Since $\mathfrak{F}_i \cap \mathfrak{F}_j = (1)$ for any $i \neq j$ then, according to Lemma 5, we conclude that $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) \cap \omega = \varnothing$ for any $i \neq j$, and so $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) \cap \omega = \varnothing$. Since $\mathfrak{H} \neq \mathfrak{F}$ then $\mathfrak{M} \neq (1)$ and according to Lemma 6, it is true that $\mathfrak{M} \in \Theta$.

Prove that $\mathfrak{H} \wedge_\Theta \mathfrak{M} = (1)$. Note that $\mathfrak{H} \wedge_\Theta \mathfrak{M} = \mathfrak{H} \cap \mathfrak{M}$. Suppose that $\mathfrak{H} \cap \mathfrak{M} \neq (1)$. Then there exists such a formation $\mathfrak{X} \subseteq \mathfrak{H} \cap \mathfrak{M}$ that $\mathfrak{X}$ is an atom of the lattice $\Theta$. In view of $\mathfrak{X} \subseteq \mathfrak{H}$ and $\mathfrak{X} \subseteq \mathfrak{M}$, we obtain $\mathfrak{X} = \mathfrak{F}_j$ for some $j \in J$ and $\mathfrak{X} = \mathfrak{F}_{i_1}$ for some $i_1 \in I_1$. Contradiction. Therefore, $\mathfrak{H} \cap \mathfrak{M} = (1)$.

Establish that $\mathfrak{H} \vee_\Theta \mathfrak{M} = \mathfrak{F}$. Indeed, since $\mathfrak{H} \cup \mathfrak{M} \subseteq \mathfrak{F}$ and $\mathfrak{F} \in \Theta$ then $\mathfrak{H} \vee_\Theta \mathfrak{M} \subseteq \mathfrak{F}$. Prove that $\mathfrak{F} \subseteq \mathfrak{H} \vee_\Theta \mathfrak{M}$. Since $\mathfrak{F} = \mathfrak{H} \oplus \mathfrak{M}$, it is sufficient to verify that $\mathfrak{H} \oplus \mathfrak{M} \subseteq \mathfrak{H} \vee_\Theta \mathfrak{M}$. Let $A \in \mathfrak{H} \oplus \mathfrak{M}$. If $A \in \mathfrak{H}$ then $A \in \mathfrak{H} \cup \mathfrak{M} \subseteq \mathfrak{H} \vee_\Theta \mathfrak{M}$. If $A \in \mathfrak{M}$ then similarly $A \in \mathfrak{H} \vee_\Theta \mathfrak{M}$. Put $A = H \times M$ where $H \in \mathfrak{H}$, $M \in \mathfrak{M}$, $H \neq 1, M \neq 1$. It means that

$$A/H \cong M \in \mathfrak{M} \subseteq \mathfrak{H} \vee_\Theta \mathfrak{M}, \quad A/M \cong H \in \mathfrak{H} \subseteq \mathfrak{H} \vee_\Theta \mathfrak{M}.$$

Since $\mathfrak{H} \vee_\Theta \mathfrak{M}$ is a formation then $A \cong A/(H \cap M) \in \mathfrak{H} \vee_\Theta \mathfrak{M}$. Hence, $\mathfrak{F} = \mathfrak{H} \oplus \mathfrak{M} \subseteq \mathfrak{H} \vee_\Theta \mathfrak{M}$ and, therefore, $\mathfrak{H} \vee_\Theta \mathfrak{M} = \mathfrak{F}$. Thus, $\Theta$ is a lattice with complements.

The theorem is proved.

## 5. Corollaries and concluding remarks

Since $\mathbb{P}FR$-functions $\delta_1$, $\delta_2$, $\delta_3$ are the *bp*-directions and $\delta_1 < \delta_2 < \delta_3$ then Theorem 1 implies the results for $\omega$-local, $\omega$-special and $\omega$-central formations.

**Corollary 1.** *The lattice of all $\omega$-local formations is modular* ([15, Theorem 4, case $n = 1$]).

**Corollary 2.** *The lattice of all $\omega$-special formations is modular.*

**Corollary 3.** *The lattice of all $\omega$-central formations is modular.*

Let $\Theta$ be an arbitrary lattice of the formations, $\mathfrak{F}_1, \mathfrak{F}_2 \in \Theta$, $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. Denote by $\mathfrak{F}_2/_\Theta\mathfrak{F}_1$ the following sublattice of the lattice $\Theta$:

$$\mathfrak{F}_2/_\Theta\mathfrak{F}_1 := \{\mathfrak{H} \in \Theta \mid \mathfrak{F}_1 \subseteq \mathfrak{H} \subseteq \mathfrak{F}_2\ \} \quad [13].$$

If $\Theta$ is a modular lattice then as follows from [1, Chapter 1, Paragraph 7], for any $\mathfrak{F}_1, \mathfrak{F}_2 \in \Theta$ the lattices $(\mathfrak{F}_1 \vee_\Theta \mathfrak{F}_2)/_\Theta\mathfrak{F}_2$ and $\mathfrak{F}_1/_\Theta(\mathfrak{F}_1 \wedge_\Theta \mathfrak{F}_2)$ are isomorphic. Therefore, according to Theorem 1, we obtain the following result.

**Corollary 4.** *Let $\delta$ be a bp-direction such that $\delta_1 \leq \delta \leq \delta_3$. Then for any formations $\mathfrak{F}_1, \mathfrak{F}_2 \in \Theta_{\omega\delta}$ the following lattice isomorphism holds*:

$$(\mathfrak{F}_1 \vee_{\Theta_{\omega\delta}} \mathfrak{F}_2)/_{\Theta_{\omega\delta}}\mathfrak{F}_2 \ \cong\ \mathfrak{F}_1/_{\Theta_{\omega\delta}}(\mathfrak{F}_1 \wedge_{\Theta_{\omega\delta}} \mathfrak{F}_2).$$

Note that a distributive lattice with complements is called a *Boolean* lattice [1]. In view of this, Theorems 2 and 3 imply the following result.

**Corollary 5.** *Let $\delta$ be a bp-direction, $(1) \neq \mathfrak{F} \in \Theta_{\omega\delta}$, $\mathfrak{F} = \oplus_{i\in I}\mathfrak{F}_i$ where $\{\mathfrak{F}_i \mid i \in I\}$ is a set of all atoms of the lattice $\Theta_{\omega\delta}(\mathfrak{F})$. If $\pi(\mathfrak{F}_i) \cap \omega \neq \varnothing$ for any $i \in I$ then $\Theta_{\omega\delta}(\mathfrak{F})$ is a Boolean lattice.*

**Remark 1.** Let $\delta$ be a $\mathbb{P}FR$-function, $f : \mathbb{P} \to \{\text{formations of groups}\}$ is a function called a $\mathbb{P}F$-function. A formation $\mathfrak{F} = (G \in \mathfrak{E} \mid G/G_{\delta(p)} \in f(p) \text{ for all } p \in \pi(G))$ is called *a fibered formation with a direction* $\delta$ and with a *satellite* $f$ [18]. According to Theorem 3 [18], for a nonempty nonidentity formation $\mathfrak{F}$, where $\pi(\mathfrak{F}) \subseteq \omega$, $\mathfrak{F}$ is a fibered formation if and only if $\mathfrak{F}$ is an $\omega$-fibered formation. Consequently, the established in the Theorems 1–3 lattice properties of $\omega$-fibered formations are valid for fibered formations of finite groups, in particular, for local, special and central formations.

**Remark 2.** According to [13, Corollary 4.2.8], the lattice of all local formations is not distributive. The author does not know if there exist conditions under which the lattice of all $\omega$-fibered (fibered) formations is a distributive lattice.

The author is grateful to M.M. Sorokina for constructive comments and valuable advices during writing this work.

**REFERENCES**


1. Birkhoff G. *Lattice Theory.* Providence, R.I.: Amer. Math. Soc., 1967, 418 p. ISBN: 0-8218-1025-1 . Translated to Russian under the title *Teoriya reshetok.* Moscow: Nauka Publ., 1984, 568 p.
2. Doerk K., Hawkes T. *Finite soluble groups.* Berlin; New York: Walter de Gruyter, 1992, 901 p. ISBN: 3-11-012892-6 .
3. Gaschütz W. Zur Theorie der endlichen auflösbaren Gruppen. *Math Z.*, 1962, vol. 80, no. 1, pp. 300–305. doi: 10.1007/BF01162386 .
4. Hartley B. On Fischer's dualization of formation theory. *Proc. London Math. Soc.*, 1969, vol. 3, no. 9. pp. 193–207. doi: 10.1112/plms/s3-19.2.193 .
5. Kamornikov S.F., Selkin M.V. *Subgroup functors and classes of finite groups.* Minsk: Belarusskaya Nauka Publ., 2003, 354 p. ISBN: 985-08-0556-0 .
6. Kamozina O.V. Non-singly-generated multiply $\omega$-fibered Fitting classes of finite groups. *Math. Notes*, 2006, vol. 79, no. 3. pp. 366–376. doi: 10.4213/mzm2709 .
7. Kamozina O.V. Algebraic lattices of multiply $\Omega$-foliated Fitting classes. *Discrete Math. Appl.*, 2006, vol. 16, no. 3, pp. 299–305. doi: 10.4213/dm53 .
8. Shemetkov L.A. *Formatsii konechnykh grupp* [Formations of finite groups]. Moscow: Nauka Publ., 1978, 272 p.
9. Shemetkov L.A., Skiba A.N. *Formations of algebraic systems.* Moscow: Nauka Publ., 1989, 255 p. ISBN: 5-02-013918-1 .
10. Skachkova (Yelovikova) Yu.A. Lattices of $\Omega$-fibered formations. *Discrete Math. Appl.*, 2002, vol. 12, no. 3, pp. 269–278. doi: 10.4213/dm243 .
11. Skachkova (Yelovikova) Yu.A. Boolean lattices of multiply $\Omega$-foliated formations. *Discrete Math. Appl.*, 2002, vol. 12, no. 5, pp. 477–482. doi: 10.4213/dm252 .
12. Skiba A.N. On local formations of the length 5. In: *Arithmetic and subgroup structure of finite groups.* Minsk: Nauka i Tekhnika Publ., 1986, pp. 149–156.
13. Skiba A.N. *Algebra of formations.* Minsk: Belarusskay Nauka Publ., 1997, 240 p. ISBN: 985-08-0078-x .
14. Skiba A.N., Shemetkov L.A. Multiply $\mathfrak{L}$-composite formations of finite groups. *Ukr. Math. J.*, 2000, vol. 52, no. 6, pp. 898–913. doi: 10.1007/BF02591784 .
15. Skiba A.N., Shemetkov L.A. Multiply $\omega$-local formations and Fitting classes of finite groups. *Siberian Adv. Math.*, 2000, vol. 10, no. 2, pp. 112–141.
16. Vasil'ev A.F., Kamornikov S.F., Semenchuk V.N. On the lattices of subgroups of finite groups. In: *Infinite groups and adjacent algebraic systems.* Kiev: Institute of Mathematics of Academy of Sciences of Ukraine, 1993, pp. 27–54.
17. Vedernikov V.A., Sorokina M.M. $\Omega$-Foliated formations and Fitting classes of finite groups. *Discrete Math. Appl.*, 2001, vol. 11, no. 5, pp. 507–527. doi: 10.4213/dm299 .
18. Vedernikov V.A., Sorokina M.M. $\omega$-Fibered formations and Fitting classes of finite groups. *Math. Notes*, 2002, vol. 71, no. 1, pp. 39–55. doi: 10.4213/dm299 .
19. Vedernikov V.A. On new types of $\omega$-fibered formations of finite groups. In: *Ukrainian Mathematical Congress – 2001. Section 1.* Kiev: Inst. Matematiki NAN Ukrainy, 2002, ISBN: 966-02-2732-9 , pp. 36–45.
20. Vorob'ov N.N. *Algebra of classes of finite groups.* Vitebsk: BSU named after P.M. Masherov, 2012, 322 p. ISBN: 978-985-517-393-0 .
21. Wielandt H. Eine Verallgemeinerung der invarianten Untergruppen. *Math Z.*, 1939, vol. 45, pp. 209–244. doi: 10.1007/BF01580283 .


Received October 5, 2020
Revised January 18, 2021
Accepted January 25, 2021

*Seraphim Pavlovich Maksakov,* Posgraduate student, Department of Mathematical Analysis, Algebra and Geometry, Bryansk State University named after Academician I.G. Petrovsky, Bryansk, 241036 Russia, e-mail: msp222@mail.ru .

Cite this article as: S. P. Maksakov. On the lattices of the $\omega$-fibered formations of finite groups, *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2021, vol. 27, no. 1, pp. 258–267 .