ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2021, Vol. 315, Suppl. 1, pp. S117–S127.

published in Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2020, Vol. 26, No. 4, pp. 126–137.

Bounds of the Nikol'skii Polynomial Constants in L^p with Gegenbauer Weight

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Received September 13, 2020; revised November 2, 2020; accepted November 9, 2020

Abstract—We study bounds and the asymptotic behavior as $n \to \infty$ of a sharp Nikol'skii constant in the inequality $\|u\|_\infty \le \mathcal{C}_\alpha(n)\|u\|_p$ for trigonometric and algebraic polynomials of degree at most n in the space L^p on $(-\pi, \pi]$ with the periodic Gegenbauer weight $|\sin x|^{2\alpha+1}$ and on $[-1, 1]$ with the algebraic Gegenbauer weight $(1-x^2)^\alpha$, respectively. We prove that $\mathcal{C}_\alpha(n) \sim \mathcal{L}_p n^{(2\alpha+2)/p}$ for $p \ge 1$ and all $\alpha \ge -1/2$, where $\mathcal{L}_p$ is a sharp Nikol'skii constant for entire functions of exponential type at most 1 in the space L^p on $\mathbb{R}$ with the power weight $|x|^{2\alpha+1}$. Moreover, we give explicit bounds of the form

$$n^{(2\alpha+2)/p}\mathcal{L}_p \le \mathcal{C}_\alpha(n) \le (n + 2s_{p,\alpha})^{(2\alpha+2)/p}\mathcal{L}_p, \quad n \ge 0,$$

from which this asymptotics follows. These bounds make it possible to refine the known estimates of the Nikol'skii constants. We consider this approach using the example of the algebraic Nikol'skii constant for $\alpha = 0$. Here we apply the characterization of the extremal polynomials from the works of D. Amir and Z. Ziegler and of V.V. Arestov and M.V. Deikalova. Our statements generalize the well-known results of S.B. Stechkin ($p = 1$) and E. Levin and D. Lubinsky ($p > 0$) in the trigonometric case for $\alpha = -1/2$ and M.I. Ganzburg in the algebraic case for $\alpha = 0$. For half-integer $\alpha = d/2 - 1$ and $p \ge 1$, our asymptotics can be derived from the asymptotics of the multidimensional Nikol'skii constant for spherical polynomials in the space L^p on the sphere $\mathbb{S}^d$ proved by F. Dai, D. Gorbachev, and S. Tikhonov. Our proof is much simpler, but it does not cover the case $p < 1$.

Keywords: Nikol'skii inequality, sharp constant, asymptotic behavior, trigonometric polynomial, algebraic polynomial, entire function of exponential type, Gegenbauer weight.

DOI: 10.1134/S0081543821060109

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