

# Recognition of the Sporadic Simple Groups $Ru$ , $HN$ , $Fi_{22}$ , $He$ , $M^cL$ , and $Co_3$ by Their Gruenberg–Kegel Graphs

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**Abstract**—The Gruenberg–Kegel graph (prime graph)  $\Gamma(G)$  of a finite group  $G$  is a graph in which the vertices are the prime divisors of the order of  $G$  and two distinct vertices  $p$  and  $q$  are adjacent if and only if  $G$  contains an element of order  $pq$ . The problem of recognition of finite groups by their Gruenberg–Kegel graph is of great interest in finite group theory. For a finite group  $G$ ,  $h_\Gamma(G)$  denotes the number of all pairwise nonisomorphic finite groups  $H$  such that  $\Gamma(H) = \Gamma(G)$  (if the set of such groups  $H$  is infinite, then  $h_\Gamma(G) = \infty$ ). A group  $G$  is called  $n$ -recognizable by its Gruenberg–Kegel graph if  $h_\Gamma(G) = n < \infty$ , recognizable by its Gruenberg–Kegel graph if  $h_\Gamma(G) = 1$ , and unrecognizable by its Gruenberg–Kegel graph if  $h_\Gamma(G) = \infty$ . We say that the problem of recognition by the Gruenberg–Kegel graph is solved for a finite group  $G$  if the value  $h_\Gamma(G)$  is found. For a finite group  $G$  unrecognizable by its Gruenberg–Kegel graph, the question of the (normal) structure of finite groups having the same Gruenberg–Kegel graph as  $G$  is also of interest. In 2003, M. Hagie investigated the structure of finite groups having the same Gruenberg–Kegel graph as some sporadic simple groups. In particular, she gave first examples of finite groups recognizable by their Gruenberg–Kegel graphs; they were the sporadic simple groups  $J_1$ ,  $M_{22}$ ,  $M_{23}$ ,  $M_{24}$ , and  $Co_2$ . However, that investigation was not completed. In 2006, A.V. Zavarnitsine established that the group  $J_4$  is recognizable by its Gruenberg–Kegel graph. The unrecognizability of the sporadic groups  $M_{12}$  and  $J_2$  by their Gruenberg–Kegel graph was known previously; it follows from the unrecognizability of these groups by their spectrum. In the present paper, we continue Hagie's study and use her results. For any sporadic simple group  $S$  isomorphic to  $Ru$ ,  $HN$ ,  $Fi_{22}$ ,  $He$ ,  $M^cL$ , or  $Co_3$ , we find all finite groups having the same Gruenberg–Kegel graph as  $S$ . Thus, for these six groups, we complete Hagie's investigation and, in particular, solve the problem of recognizability by the Gruenberg–Kegel graph.

**Keywords:** finite group, simple group, sporadic group, spectrum, Gruenberg–Kegel graph, recognition by the Gruenberg–Kegel graph.

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