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BEST RESTRICTED APPROXIMATION OF SMOOTH FUNCTION CLASSES[1]

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We first discuss the relative Kolmogorov $n-$widths of classes of smooth 2π-periodic functions for which the modulus of continuity of their r-th derivatives does not exceed a given modulus of continuity, and then discuss the best restricted approximation of classes of smooth bounded functions defined on the real axis $\mathbb{R}$ such that the modulus of continuity of their r-th derivatives does not exceed a given modulus of continuity by taking the classes of the entire functions of exponential type as approximation tools. Asymptotic results are obtained for these two problems.

Keywords: modulus of continuity, best restricted approximation, average width.

Юнпин Лю, Гуйцяо Сюй, Цзе Чжан. Наилучшая аппроксимация с ограничениями для классов гладких функций.

Обсуждаются относительные n-поперечники по Колмогорову для классов гладких 2π-периодических функций, определяемых модулем непрерывности, а также наилучшая аппроксимация с ограничениями целыми функциями экспоненциального типа для классов гладких ограниченных функций, определенных на числовой оси $\mathbb{R}$ и таких, что модуль непрерывности их r-й производной не превосходит заданного модуля непрерывности. Для этих двух задач получены асимптотические результаты.

Ключевые слова: модуль непрерывности, наилучшая аппроксимация с ограничениями, средний поперечник.

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1. Introduction

Denote by $\mathbb{Z}_+$ the set of all nonnegative integers, $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$, by $\mathbb{R}$ the set of all real numbers or the real axis, and by $\mathbb{C}$ the set of all complex numbers or the complex plane.

Let $P_r(\lambda)$ is an algebraic polynomial of degree $r \in \mathbb{N}$ in the form

$$P_r(\lambda) = (\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_r), \tag{1.1}$$

where $\lambda_k := \alpha_k + i\beta_k$, $\alpha_k \in \mathbb{R}$, $\beta_k \in \mathbb{R}$, $k = 1, 2, \cdots, r$. Denote by

$$P_r(D) := \Big(\frac{d}{dx} - \lambda_1 I\Big) \cdot \ldots \cdot \Big(\frac{d}{dx} - \lambda_r I\Big), \quad D := \frac{d}{dx},$$

the linear differential operator of order r with respect to P_r, where I is the identity operator.

For the interval $\mathbb{R}$ (or $\mathbb{T} = [0, 2\pi]$) and $1 \le p \le +\infty$, let $L_p = L_p(\mathbb{R})$ (or $\tilde{L}_p = \tilde{L}_p(\mathbb{T})$) denote the Banach space of (or 2π-periodic) functions $f : \mathbb{R} \to \mathbb{C}$ to be $p-$power integrable on $\mathbb{R}$ (or on $\mathbb{T}$) with the usual L_p-norm $\|f\|_{L_p}$(or $\|f\|_{\tilde{L}_p}$).

For a nonnegative integer $r \in \mathbb{Z}_+$, denote by C^r (or $\tilde{C}^r$) the collection of all functions f for which the r-order derivatives $f^{(r)}$ $(f^{(0)} = f)$ are uniformly continuous (or 2π-periodic and

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continuous). Denote by $W_p^{P_r}$ (or $\tilde{W}_p^{P_r}$) the collection of functions $f \in L_p \cap C^r$ (or $f \in \tilde{C}^r$) for which have the locally absolutely continuous derivatives up to the $(r-1)$st order and $\|P_r(D)f\|_{L_p} \le 1$ (or $\|P_r(D)f\|_{\tilde{L}_p} \le 1$). Specially, when $P_r(\lambda) = \lambda^r$, write W_p^r (or $\tilde{W}_p^r$) instead of $W_p^{P_r}$ (or $\tilde{W}_p^{P_r}$). Let ω be a modulus of continuity. Set

$$W^r H^\omega = \{f \in C^r : f^{(r)} \in H^\omega\}, \quad \tilde{W}^r H^\omega = \{f \in \tilde{C}^r : f^{(r)} \in H^\omega\},$$

where $f^{(r)} \in H^\omega$ means

$$\omega(f^{(r)}, t) = \sup\{|f^{(r)}(x) - f^{(r)}(y)| : x, y \in \mathbb{R}, |x-y| \le t\} \le \omega(t), \quad t \ge 0.$$

When $\omega(t) = t^\alpha, 0 < \alpha \le 1$, we write $W^r H^\alpha$ (or $\tilde{W}^r H^\alpha$) instead of $W^r H^\omega$ (or $\tilde{W}^r H^\omega$).

Let E_σ, $\sigma \ge 0$, denote the class of entire functions of exponential type σ, that is, $f \in E_\sigma$ if and only if $f : \mathbb{C} \to \mathbb{C}$ is an entire function and for $\forall \epsilon > 0$, $\exists A_\epsilon > 0$ such that

$$|f(z)| \le A_\epsilon \exp((\sigma + \epsilon)|z|), \quad z = x + iy \in \mathbb{C}.$$

Denote by $E_{\sigma,p} := E_{\sigma,p}$, $1 \le p \le +\infty$, the collection of all $f \in E_\sigma$ with $f|_{\mathbb{R}} \in L_p$. Here the notation $F(\sigma) \asymp \sigma^s$ $(s < 0)$ means that there exist two positive real numbers $C, D > 0$ independent of σ for which $D\sigma^s \le F(\sigma) \le C\sigma^s$ for all sufficiently large σ (with the analogous meaning for $n^s \ll F(\sigma) \ll n^s$, $F(n) \asymp n^s$, etc).

Let W and V be two nonempty subsets of a normed linear space X endowed with the norm $\|\cdot\|_X$. Denote by

$$E(W, V)_X := \sup_{w \in W} \inf_{v \in V} \|w - v\|_X$$

the deviation (or approximation) of W from V in the space X. In 1984, Konovalov [7] raised the problem to consider the relative n-width of W related to V in the space X is given by

$$d_n(W, V)_X := \inf_L E(W, L \cap V)_X,$$

where the infimum is taken over all n-dimensional subspaces L of X. When $V = X$, $d_n(W, V)_X = d_n(W)_X$ is the usual n-width, in the sense of Kolmogorov, W in X. V. N. Konovalov in [7] proved that

$$d_n(\tilde{W}_\infty^r, \tilde{W}_\infty^r)_\infty \asymp n^{-2}, \quad r \ge 3.$$

V. F. Babenko in [2] showed

$$d_n(\tilde{W}_1^r, \tilde{W}_1^r)_1 \asymp n^{-2}, \quad r \ge 3.$$

V. M. Tikhomirov in [22] generalized the above result in [7] from the positive integer $r \ge 3$ to the positive real number α through a simpler proof.

For $1 \le q \le \infty$, $r \in \mathbb{N}$, V. N. Konovalov in [8; 9] proved

$$\begin{aligned} &d_n(\tilde{W}_\infty^r, \tilde{W}_\infty^r)_q \asymp n^{-\min\{r,2\}}, \\ &d_n(\tilde{W}_1^r, \tilde{W}_1^r)_q \asymp n^{-\min\{r-1+\frac{1}{q}, 2\}}, \quad (r, q) \ne (1, \infty) \\ &d_n(\tilde{W}_2^r, \tilde{W}_2^r)_q \asymp n^{-\min\{r-\frac{1}{2}+\frac{1}{q}, r\}}. \end{aligned}$$

Later, Wei Yang [25] considered the relative n-widths of two kinds of periodic convolution classes whose convolution kernels: NCVD-kernel and B-kernel, and obtained the similar asymptotic estimates.

Tikhomirov [23] introduced the concept of the average dimension and Magaril-Il'yaev [16] proposed the concept of the average width. More general statement was formulated by Professor Yongsheng Sun in [21]. For the special case L_p, $p \in [1, \infty]$, we state the definitions of the average dimension and average width as follows.

Let L be a subspace of L_p and $B_L := \{x \in L \colon \|x\|_{L_p} \le 1\}$. For any $\epsilon > 0, a > 0$, let

$$N(\epsilon, a) := \min \big\{ n \in \mathbb{Z}_+ \colon \exists \text{ linear subspace } M \subset L_p[-a,a] \\ \text{with } dim(M) = n,\ s.t.\ E(B_L|_{[-a,a]}, M)_{L_p[-a,a]} < \epsilon \big\}.$$

It is easy to see that $N(\epsilon, a)$ is increasing in variable a and decreasing in variable ϵ. The number

$$\overline{dim}(L; L_p) := \lim_{\epsilon \to 0^+} \liminf_{a \to +\infty} \frac{N(\epsilon, a)}{2a}$$

is said to be the average dimension of L in L_p. Specially, $\overline{dim}(E_{\sigma,p}; L_p) = \dfrac{\sigma}{\pi}$ (see [16]).

For $\sigma > 0$, the quantity

$$\bar{d}_\sigma(W)_{L_p} := \inf \big\{ E(W, L)_{L_p} \colon L \text{ is a subspace of } L_p \text{ with } \overline{dim}(L; L_p) \le \sigma \big\}$$

is called the average Kolmogorov $\sigma-$width of W in the space L_p.

On the average width of Sobolev classes $W_p^{P_r}(\mathbb{R})$ (or Sobolev–Wiener classes $W_{p,q}^r(\mathbb{R})$) in the metric $L_p(\mathbb{R})$ (or other classes of smooth functions on $\mathbb{R}$ or $\mathbb{R}^d (d > 1)$), many exact results are obtained by Magaril–Il'yaev [17; 18], Dirong Chen [3], Yongping Liu [13], Heping Wang [20], Yanjie Jiang [6], Guiqiao Xu [24], etc.

Combining the ideas of Magaril–Il'yaev[16] and Konovalov[7], Liu and Xiao [14] introduced the problem to consider the quantity

$$\bar{d}_\sigma(W, V)_{L_p} := \inf_L E(W, L \cap V)_{L_p},$$

where the infimum is taken over all subspaces L of L_p with $\overline{dim}(L; L_p) \le \sigma$, and call it to be the relative average Kolmogorov $\sigma-$width of W related to V in the space L_p. Obviously, when $V = L_p$, $\bar{d}_\sigma(W, V)_{L_p} = \bar{d}_\sigma(W)_{L_p}$ is the average Kolmogorov σ-width of W in the space L_p. In [14], Liu and Xiao gave some exact results on some classes of functions in $L_2(\mathbb{R}^d)$ as follows.

For $\alpha > 0$, set

$$\mathfrak{R}_2^\alpha(\mathbb{R}^d) = \left\{ f \in L_2(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |y|^{2\alpha} |\hat{f}(y)|^2 dy \le 1 \right\},$$

$$\mathfrak{B}_2^\alpha(\mathbb{R}^d) = \left\{ f \in L_2(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} \left(1 + |y|^2\right)^\alpha |\hat{f}(y)|^2 dy \le 1 \right\}.$$

Where $\hat{f}$ denotes the Fourier transform of f, and $|y|$ denotes the length of a vector $y \in \mathbb{R}^d$ defined by $|y| = \sqrt{(y, y)}$, while (x, y) denotes the inner product of two vectors $x, y \in \mathbb{R}^d$.

Theorem 1 [14].

$$\overline{d}_\sigma\big(\mathfrak{R}_2^\alpha(\mathbb{R}^d), \mathfrak{R}_2^\alpha(\mathbb{R}^d)\big)_{L_2(\mathbb{R}^d)} = \overline{d}_\sigma\big(\mathfrak{R}_2^\alpha(\mathbb{R}^d)\big)_{L_2(\mathbb{R}^d)} = (\rho(\sigma))^{-\alpha};$$

$$\overline{d}_\sigma\big(\mathfrak{R}_2^\alpha(\mathbb{R}^d), M\mathfrak{R}_2^\alpha(\mathbb{R}^d)\big)_{L_2(\mathbb{R}^d)} = \infty, \quad 0 < M < 1.$$

$$\overline{d}_\sigma\big(\mathfrak{B}_2^\alpha(\mathbb{R}^d), M\mathfrak{B}_2^\alpha(\mathbb{R}^d)\big)_{L_2(\mathbb{R}^d)} = \overline{d}_\sigma\big(\mathfrak{B}_2^\alpha(\mathbb{R}^d)\big)_{L_2(\mathbb{R}^d)} = 1 - M_0, \quad M \ge M_0;$$

$$\overline{d}\big(\mathfrak{R}_2^\alpha(\mathbb{R}^d), M\mathfrak{R}_2^\alpha(\mathbb{R}^d)\big)_{L_2(\mathbb{R}^d)} = 1 - M, \quad 0 < M < M_0.$$

Where $M_0 := 1 - \left(1 + (\rho(\sigma))^2\right)^{-\alpha/2}$, $\rho(\sigma) = \sqrt{4\pi}\left(\Gamma\left(\frac{d}{2} + 1\right)\sigma\right)^{1/d}$. $SB^2_{\rho(\sigma)}(\mathbb{R}^d)$, *the collection of the entire functions of spherical exponential type* $\sigma \ge 0$, *which as functions of the real vector* $x \in \mathbb{R}^d$ *lie in* $L_2(\mathbb{R}^d)$, *is an optimal space.*

On the exact order of relative average width $\bar{d}_\sigma\left(W_p^{P_r}, W_p^{P_r}\right)_{L_p}$ and the best restricted approximation $E\left(W_p^{P_r}, E_\sigma \cap W_p^{P_r}\right)_{L_p}$ for $p=1,2,\infty$, Ling and Liu obtained the following results.

Theorem 2 [12]. *Let P_r be a polynomial in the form of (1.1) with $r \geq 1$ and $\sigma_0 := \inf\{\sigma > 0 : P_r(i\lambda) \neq 0, \forall |\lambda| > \sigma\}$. Then $\bar{d}_\sigma\big(W_2^{P_r}, W_2^{P_r}\big)_{L_2} = E\big(W_2^{P_r}, E_\sigma \cap W_2^{P_r}\big)_{L_2} = 2\pi \max_{|y| \geq \sigma} \dfrac{1}{|P_r(iy)|}$ for all $\sigma > \sigma_0$.*

It is worth to mention that the above theorem has been proved essentially in [14].

For $r \in \mathbb{N}$, let $P_r(\lambda)$ be an algebraic polynomial of degree r in the form of

$$P_r(\lambda) = \lambda^s \prod_{j=1}^{r-s} (\lambda - \lambda_j) \tag{1.2}$$

where $s \leq r$ and $s \in \{0,1,2\}$, and (Non pure imaginary) $\lambda_j \notin i\mathbb{R} := \{ix\colon x \in \mathbb{R}\}$, $j = 1, 2, \cdots, r-s$. When $r = s \in \{0,1,2\}$, $P_s(\lambda) = \lambda^s$.

Theorem 3 [12]. *Let $P_r(\lambda)$ be an algebraic polynomial of degree r in the above form. Then*

$$\bar{d}_\sigma\left(W_p^{P_r}, W_p^{P_r}\right)_{L_p} \asymp E\left(W_p^{P_r}, E_\sigma \cap W_p^{P_r}\right)_{L_p} \asymp \sigma^{-r}, \quad r = 1,2, \quad 1 \leq p \leq +\infty;$$

$$E\left(W_p^{P_r}, E_\sigma \cap W_p^{P_r}\right)_{L_p} \ll \sigma^{-\min(2,r)}, \quad 1 \leq p \leq +\infty, \quad r \in \mathbb{N};$$

$$E\left(W_p^{P_r}, E_\sigma \cap W_p^{P_r}\right)_{L_p} \gg \sigma^{-\min(2,r)}, \quad p = 1, +\infty, \quad r \in \mathbb{N}.$$

Remark. Theorem 3 shows that $\bar{d}_\sigma\left(W_p^{P_r}, W_p^{P_r}\right)_{L_p} \asymp \bar{d}_\sigma\left(W_p^{P_r}\right)_{L_p} \asymp \sigma^{-r}$, $r = 1,2$, $1 \leq p \leq +\infty$; $E\left(W_p^{P_r}, E_\sigma \cap W_p^{P_r}\right)_{L_p} \asymp \sigma^{-\min(2,r)}$, $p = 1, +\infty$, $r \in \mathbb{N}$.

We conjecture that, under the assumption of Theorem 3, it also holds that

$$\bar{d}_\sigma\left(W_p^{P_r}, W_p^{P_r}\right)_{L_p} \asymp \sigma^{-\min(2,r)} \quad \text{for} \quad p \in \{1, \infty\}.$$

2. Our main results

Theorem 4. *Let $1 \leq q \leq +\infty$ and the modulus of continuity ω be concave. Then*

$$d_{2n-1}(\tilde{W}^r H^\omega, \tilde{W}^r H^\omega)_{\tilde{L}_q} \asymp E(\tilde{W}^r H^\omega, T_n \cap \tilde{W}^r H^\omega)_{\tilde{L}_q} \asymp \begin{cases} n^{-2}, & r \geq 2, \\ n^{-r}\omega\Big(\dfrac{1}{n}\Big), & r = 0,1. \end{cases} \tag{2.1}$$

Where T_n denotes the linear manifold of trigonometric polynomials

$$\frac{a_0}{2} + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx)$$

of degree n.

Corollary 1. *Let $\alpha \in (0,1]$ and $1 \leq q \leq +\infty$. Then*

$$d_{2n-1}(\tilde{W}^r H^\alpha, \tilde{W}^r H^\alpha)_{\tilde{L}_q} \asymp E(\tilde{W}^r H^\alpha, T_n \cap \tilde{W}^r H^\alpha)_{\tilde{L}_q} \asymp n^{-\min\{2, r+\alpha\}}.$$

Theorem 5. *Let the modulus of continuity ω be concave and $\sigma \geq 2$. Then*

$$E(W^r H^\omega, E_\sigma \cap W^r H^\omega)_{L_\infty} \asymp \begin{cases} \sigma^{-2}, & r \geq 2, \\ \sigma^{-r}\omega\Big(\dfrac{1}{\sigma}\Big), & r = 0,1. \end{cases}$$

Corollary 2. *Let $0 < \alpha \le 1$ and $\sigma \ge 2$. Then*

$$E(W^r H^\alpha, E_\sigma \cap W^r H^\alpha)_{L_\infty} \asymp \sigma^{-\min\{2, r+\alpha\}}.$$

We conjecture that, under the assumption of Theorem 5, it also holds that

$$\bar{d}_\sigma\left(W^r H^\omega, W^r H^\omega\right)_{L_\infty} \asymp \begin{cases} \sigma^{-2}, & r \ge 2, \\ \sigma^{-r}\omega\Big(\dfrac{1}{\sigma}\Big), & r = 0, 1. \end{cases}$$

3. Proofs of the Theorem 4 and 5

The proof of Theorem 4. The upper estimate. For $n \in \mathbb{N}$, let

$$(J_n f)(x) = \int\limits_{\mathbb{T}} f(x+t) k_n(t) dt.$$

Where $k_n(t) = L_{n'}(t)\Big(n' = \Big[\dfrac{n+1}{2}\Big]\Big)$, while $L_n(t) = \lambda_n^{-1}\left(\dfrac{\sin\dfrac{nt}{2}}{\sin\dfrac{t}{2}}\right)^4$, is the Jackson kernel. Here λ_n is determined by the equality $\int_{\mathbb{T}} L_n(t)dt = 1$. It is well-known that $J_n(f)$ is a trigonometric polynomial of degree $< n$.

If $f \in \tilde{W}^r H^\omega$, it is easy to see that $(J_n f)^{(r)} \in \tilde{W}^r H^\omega$. From Jackson theorem (see [4, Theorem 2.2 in Ch. 7]), there is some absolute constant C such that

$$\|J_n f - f\|_{\tilde{L}_q} \le C\omega_2\Big(f, \frac{1}{n}\Big)_{\tilde{L}_q}, \tag{3.1}$$

where

$$\omega_2\left(f, t\right)_{\tilde{L}_q} = \sup_{|h| \le t} \|f(\cdot + 2h) - 2f(\cdot + h) + f(\cdot)\|_{\tilde{L}_q},\ 0 \le t < +\infty.$$

It is easy to see that when $r = 0, 1$, $\omega_2\Big(f, \dfrac{1}{n}\Big)_{\tilde{L}_q} \ll \dfrac{1}{n^r}\omega\Big(\dfrac{1}{n}\Big)$, when $r \ge 2$, $\omega_2\Big(f, \dfrac{1}{n}\Big)_{\tilde{L}_q} \ll \dfrac{1}{n^2}\|f''\|_{\tilde{L}_q}$ and there exists some absolute constant C_r dependent only on r such that

$$\|f''\|_{\tilde{L}_q} \le C_r. \tag{3.2}$$

Thus, by these discussions, we obtain

$$d_{2n+1}(\tilde{W}^r H^\omega, \tilde{W}^r H^\omega)_{\tilde{L}_q} \le E(\tilde{W}^r H^\omega, T_n \cap \tilde{W}^r H^\omega)_{\tilde{L}_q} \ll \begin{cases} n^{-2}, & r \ge 2, \\ n^{-r}\omega\Big(\dfrac{1}{n}\Big), & r = 0, 1. \end{cases} \tag{3.3}$$

Next, we prove the lower estimate. When $r = 0, 1$, since $d_n(\tilde{W}^r H^\omega)_{\tilde{L}_q} \le d_n(\tilde{W}^r H^\omega, \tilde{W}^r H^\omega)_{\tilde{L}_q}$, then (2.1) is verified by (3.3) and the following lemma.

Lemma 1 [15]. *Let $1 \le q \le +\infty$. Then*

$$\frac{1}{n^r}\omega\Big(\frac{1}{n}\Big) \ll d_n(\tilde{W}^r H^\omega)_{\tilde{L}_q}, \quad r \in \mathbb{Z}_+. \tag{3.4}$$

When $r \ge 2$, since

$$d_{2n-1}(\tilde{W}^r H^\omega, \tilde{W}^r H^\omega)_{\tilde{L}_q} \ge d_{2n}(\tilde{W}^r H^\omega, \tilde{W}^r H^\omega)_{\tilde{L}_q} \ge d_{2n}(\tilde{W}^r H^\omega, \tilde{W}^r H^\omega)_{\tilde{L}_1}(2\pi)^{-1/q'},$$

it is sufficient to prove that

$$d_{2n}(\tilde{W}^r H^\omega, \tilde{W}^r H^\omega)_{\tilde{L}_1} \gg n^{-2}. \tag{3.5}$$

To prove (3.5), we need the standard functions $f_{n,r}$ and $\varphi_{n,r}$ and their properties. Let $f_{n,r}$ denote the standard function which realized many extremal properties of the functions from $\tilde{W}^r H^\omega$ for the concave modulus of continuity ω defined by the following way. Set

$$f_{n,0}(x) = f_{n,0}(\omega, t) = \begin{cases} \frac{1}{2}\omega(2x), & 0 \le x \le \frac{\pi}{2n}, \\ \frac{1}{2}\omega\left[2\left(\frac{\pi}{n} - x\right)\right], & \frac{\pi}{2n} \le x \le \frac{\pi}{n}, \end{cases}$$

and $f_{n,0}(x) = -f_{n,0}\Big(x - \dfrac{\pi}{n}\Big), \dfrac{\pi}{n} \le x \le \dfrac{2\pi}{n}$, and $f_{n,0}(x)$ is a $\dfrac{2\pi}{n}$-periodic function. In addition, $\gamma_{n,r} = \dfrac{\pi(1-(-1)^r)}{4n}$. Further, let $f_{n,r}$ be the r-th 2π periodic integral of $f_{n,0}$ with zero mean value on the period interval $[a, a+2\pi], a \in \mathbb{R}$, i.e.,

$$f_{n,r}(x) = \int\limits_{\gamma_{n,r}}^{x} f_{n,r-1}(t)dt, \quad r \in \mathbb{N}.$$

Another standard function $\varphi_{n,r}$ is the r-th 2π periodic integral of $\varphi_{n,0}(x) = \operatorname{sgn}\sin nx$ with zero mean value on the period interval $[a, a+2\pi], a \in \mathbb{R}$. When $n = 1$, we simply write f_r, φ_r, γ_r in stead of $f_{1,r}$, $\varphi_{1,r}, \gamma_{1,r}$. Many properties of the standard functions $f_{n,r}$ and $\varphi_{n,r}$ may be found in Korniechuk's books [10;11]. To read the article with ease, we list some properties of f_r and φ_r which will be used in the next proof. First,

$$\operatorname{sgn} f_0(x) = \varphi_0(x) = \operatorname{sgn}\sin x = \frac{4}{\pi}\sum_{v=0}^{+\infty}\frac{\sin(2v+1)x}{2v+1}. \tag{3.6}$$

On the periodic interval $[\gamma_r, \gamma_r + 2\pi]$, the two functions f_r and φ_r have only three simple zeros $\gamma_r, \gamma_r + \pi, \gamma_r + 2\pi$ and have only two extremal points $\gamma_r + \dfrac{\pi}{2}$ and $\gamma_r + \dfrac{3\pi}{2}$. Their absolute value functions $|f_r|$ and $|\varphi_r|$ are concave on the intervals $[\gamma_r, \gamma_r + \pi]$ and $[\gamma_r + \pi, \gamma_r + 2\pi]$, respectively.

To prove Theorem 4, similar to the four steps of Tikhomirov in his paper [22], we also need several lemmas as follows.

Lemma 2 [10, Proposition 2.5.2]. *Let $F \subset L_p[a,b]\,(1 \le p < \infty)$ be a convex and closed subset. Then for any $f \in L_p[a,b]$, and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have*

$$e(f,F)_{L_p[a,b]} = \inf_{g\in F}\|f-g\|_{L_p[a,b]} = \sup_{\|g\|_{L_{p'}[a,b]}\le 1}\left\{\int\limits_a^b f(t)g(t)dt - \sup_{u\in F}\int\limits_a^b u(t)g(t)dt\right\}.$$

The following lemma belongs to Ismagilov(1968).

Lemma 3 [5; 15, Theorem 4.7 in Ch. 13]. *Let $\psi \in L_2(\mathbb{T})$ be some fixed function with mean value zero, and its Fourier series can be expressed as follows $\psi(t) = \sum\limits_{k=1}^{\infty}(a_k\cos kt + b_k \sin kt)$. Denote by $k(\psi)$ be the subset of $L_2(\mathbb{T})$ formed by the translates of $\psi(\cdot)$, that is*

$$k(w) = \{T_\tau\psi(\cdot)\}_{\tau\in\mathbb{T}}, \quad T_\tau\psi(t) = \psi(t+\tau).$$

Then, for $n \in \mathbb{N}$, $d_{2n}(k(\psi), L_2(\mathbb{T})) = \Big(\pi\sum\limits_{k=n+1}^{\infty} c_k^{*2}\Big)^{1/2}$, where c_k^* denote the numbers $c_k = (a_k^2 + b_k^2)^{1/2}$ arranged in non-increasing order.

The following lemma is called Neyman–Pearson lemma.

Lemma 4 [22]. *Let $y(\cdot) \in C(\triangle)$, $y(t) \geq 0$, for any $t \in \triangle = [t_0, t_1]$,*

$$X = \left\{ x(\cdot) \in L_1(\triangle) : 0 \leq x(t) \leq A, \text{ a.e.}, \int\limits_{\triangle} x(t)dt \geq B \right\}.$$

Then

$$\int\limits_{\triangle} x(t)y(t)dt \geq A \int\limits_{D(A,B)} y(t)dt, \ \forall x(\cdot) \in X,$$

where $D(A,B) = \{t \,|\, 0 \leq y(t) \leq C(A,B)\}$, while the constant $C(A,B)$ is chosen so as to have $\displaystyle\int_{D(A,B)} dt = \frac{B}{A}$.

For a fixed arbitrary subspace $L^{2n} \subset \tilde{L}_1$, using the dual theorem of the best approximation by convex set, we find that for each fixed $\tau \in [0, 2\pi]$ the function $T_\tau f_r$ there holds the following inequality

$$E(k(f_r), L^{2n} \cap \tilde{W}^r H^\omega)_{\tilde{L}_1} = \sup_{\tau \in \mathbb{R}} e(T_\tau f_r, L^{2n} \cap \tilde{W}^r H^\omega)_{\tilde{L}_1}$$

$$\geq \sup_{\tau \in \mathbb{R}} \left\{ \int\limits_0^{2\pi} f_r(t+\tau)\mathrm{sgn} f_r(t+\tau)dt - \sup_{h \in L^{2n} \cap \tilde{W}^r H^\omega} \int\limits_0^{2\pi} h(t)\mathrm{sgn} f_r(t+\tau)dt \right\}.$$

Let $\varepsilon_r = (-1)^{(r+1)/2}$ for odd r and $(-1)^{r/2}$ for even r. It is easy to verify that the standard functions φ_r and f_r have the following sign properties

$$\mathrm{sgn} f_r(t) = \mathrm{sgn}\varphi_r(t) = \varepsilon_r \mathrm{sgn} f_0\,(t+\gamma_r) = \varepsilon_r \mathrm{sgn}\varphi_0\,(t+\gamma_r) = \varepsilon_r \varphi_0\,(t+\gamma_r) = \varepsilon_r \mathrm{sgn}\sin(t+\gamma_r).$$

Hence, we obtain that $(-1)^r \varepsilon_r \mathrm{sgn}\varphi_r(t+\gamma_r) = \varphi_0(t)$. For $h \in L^{2n} \cap \tilde{W}^r H^\omega$, using the integration by parts for r times, we may see that there hold following equalities

$$\int\limits_0^{2\pi} f_r(t+\tau)\mathrm{sgn} f_r(t+\tau)dt = \varepsilon_r \int\limits_0^{2\pi} f_r(t)\varphi_0(t+\gamma_r)dt, \tag{3.7}$$

$$\int\limits_0^{2\pi} h(t)\mathrm{sgn} f_r(t+\tau)dt = (-1)^r \varepsilon_r \int\limits_0^{2\pi} h^{(r)}(t-\tau)\varphi_r(t+\gamma_r)dt. \tag{3.8}$$

Set

$$H_\tau(t) = \frac{1}{4}\left\{ h^{(r)}(t-\tau) - h^{(r)}(-t-\tau) + h^{(r)}(\pi-t-\tau) - h^{(r)}(\pi+t-\tau) \right\}.$$

Then it is easy to verify that the 2π-periodic function H_τ is odd and satisfies

$$H_\tau(\pi - t) = H_\tau(t), \quad x \in [0, \pi]; \quad H_\tau(t) = -H_\tau(t-\pi), \quad t \in [\pi, 2\pi];$$

$$|H_\tau(t)| \leq |f_0(t)|, \quad 0 \leq t \leq 2\pi. \tag{3.9}$$

Here, we show the proof of the last inequality. Since the 2π-periodic and continuous function $h \in \tilde{W}^r H^\omega$, then, when $0 \leq t \leq \pi/2$, by $|(t-\tau) - (-t-\tau)| = |(\pi+t-\tau) - (\pi-t-\tau)| = 2t$, we see that there are following inequalities

$$|H_\tau(t)| \leq \frac{1}{4}\left\{ |h^{(r)}(t-\tau) - h^{(r)}(-t-\tau)| + |h^{(r)}(\pi-t-\tau) - h^{(r)}(\pi+t-\tau)| \right\} \leq \frac{1}{2}w(2t).$$

When $\pi/2 \le t \le \pi$, by $|(\pi - t - \tau) - (-\pi - t - \tau)| = |(t - \tau) - (2\pi - t - \tau)| = 2\pi - 2t$, we see that there are following inequalities

$$|H_\tau(t)| = |H_\tau(\pi - t)| \le \frac{1}{4}\left\{|h^{(r)}(\pi - t - \tau) - h^{(r)}(-\pi + t - \tau)| + |h^{(r)}(t - \tau) - h^{(r)}(2\pi - t - \tau)|\right\}$$
$$\le \frac{1}{2}w(2(\pi - t)).$$

Thus, by above discussion and using the definition of f_0 and the fact that $H_\tau(t) = -H_\tau(t - \pi)$ for $t \in [\pi, 2\pi]$, we obtain (3.9).

Hence, using the properties of the standard function φ_r, there is the following equality

$$\int_0^{2\pi} h^{(r)}(t - \tau)\varphi_r(t + \gamma_r)dt = \int_0^{2\pi} H_\tau(t)\varphi_r(t + \gamma_r)dt. \tag{3.10}$$

Define a sequence $\{H_{\tau,r}\}_{r=0}^{+\infty}$ of 2π-periodic functions as follows

$$H_{\tau,0}(t) = H_\tau(t), \quad H_{\tau,r}(t) = \int_{\gamma_r}^{t} H_{\tau,r-1}(t)dt, \quad r \in \mathbb{N}.$$

Hence, $H_{\tau,r}(\gamma_r) = H_{\tau,r}(\gamma_r + \pi) = H_{\tau,r}(\gamma_r + 2\pi) = 0$.

Next, we can verify that

$$|H_{\tau,r}(t)| \le |f_r(t)|, \quad t \in [0, 2\pi]. \tag{3.11}$$

Using proof by contradiction. When r is even, suppose that there a point $x_0 \in (0, 2\pi), x_0 \ne \pi$, such that $|H_{\tau,r}(x_0)| > |f_r(x_0)|$. Let $\lambda = \dfrac{f_r(x_0)}{H_{\tau,r}(x_0)}, |\lambda| < 1$, and set $\phi(t) = f_r(t) - \lambda H_{\tau,r}(t)$. Then, $\phi(x_0) = \phi(0) = \phi(\pi) = \phi(2\pi) = 0$. Using the Rolle's theorem for r times, we will see that $\phi^{(r)}(t) = f_0(t) - \lambda H_\tau(t)$ has at least 4 zeros on some closed periodic interval. In fact, without loss of generality, suppose that $0 < x_0 < \pi$. Using the Rolle's theorem on the 2π-periodic function ϕ, we see that ϕ' has at least 4 zeros on a closed periodic interval which may be written as $x_j^{(1)}, j = 1, 2, 3, 4$, satisfying

$$0 < x_1^{(1)} < x_0 < x_2^{(1)} < \pi < x_3^{(1)} < 2\pi < x_4^{(1)} = x_1^{(1)} + 2\pi.$$

By induction, $\phi^{(r)}$ has at least 4 zeros on a closed periodic interval which these zeros may be written as $x_j^{(r)}, j = 1, 2, 3, 4$, satisfying

$$x_1^{(r)} < x_2^{(r)} < x_3^{(r)} < x_4^{(r)} = x_1^{(r)} + 2\pi.$$

The so-called closed periodic interval may be chosen as $\left[x_1^{(r)}, x_1^{(r)} + 2\pi\right]$. However, the fact

$$|\lambda H_{\tau,r}^{(r)}(t)| = |\lambda H_\tau(t)| < |f_0(t)|, \quad t \in (0, 2\pi), \quad t \ne \pi,$$

shows that the function $f_0(t) - \lambda H_\tau(t)$ has only three zeros on the closed interval $[0, 2\pi]$:$0, \pi, 2\pi$. Then, $f_0(t) - \lambda H_\tau(t)$ has at most 3 zeros on the closed periodic interval $\left[x_1^{(r)}, x_1^{(r)} + 2\pi\right]$. This produces a contradiction which shows that (3.11) is true. When r is odd, the proof of (3.11) is similar.

In the right side of equality (3.10), using the integration by parts for r times again, we obtain the following equality

$$\int_0^{2\pi} H_\tau(t)\varphi_r(t + \gamma_r)dt = (-1)^r \int_0^{2\pi} H_{\tau,r}(t)\varphi_0(t + \gamma_r)dt. \tag{3.12}$$

Combination of (3.8) to (3.12) gives

$$\int_0^{2\pi} h(t)\mathrm{sgn} f_r(t+\tau)dt = \int_0^{2\pi} f_r(t)\frac{H_{\tau,r}(t)}{|f_r(t)|}dt.$$

Write

$$x_\tau(t) = \varepsilon_r\varphi_0(t+\gamma_r) - \frac{H_{\tau,r}(t)}{|f_r(t)|},\ y(t) = f_r(t), t\in[0,2\pi].$$

Since the function values of $\varepsilon_r\varphi_0(t+\gamma_r)$ are 1 and -1 except $t = 0, \pi, 2\pi$ on the interval $[0, 2\pi]$, and the functions $\left|\frac{H_{\tau,r}(t)}{|f_r(t)|}\right| \le 1, a.e.\, t\in[0,2\pi]$, then

$$\mathrm{sgn} x_\tau(t) = \mathrm{sgn}\varphi_r(t+\gamma_r), \quad |x_\tau(t)|\le 2,\ a.e.\, t\in[0,2\pi],$$

and hence $x_\tau(t)y_\tau(t) = |x_\tau(t)||y_\tau(t)|$, $a.e.\, t\in[0,2\pi]$.

By a similar discussion as in [8] and [9], we can verify that there is a point τ_0 such that

$$\int_0^{2\pi} x_{\tau_0}(t)y(t)dt = \int_0^{2\pi} |x_{\tau_0}(t)||y(t)|dt \ge \frac{c}{n^2} \tag{3.13}$$

for some absolutely positive constant c dependent only on r. In fact, because

$$\varepsilon_r\varphi_0(t+\gamma_r) = \frac{4}{\pi}\sum_{v=0}^{+\infty}\frac{\varepsilon_r\sin(2v+1)(t+\gamma_r)}{2v+1}, \quad t\in[0,2\pi],$$

and h will be taken over a subset $L^{2n}\cap\tilde{W}^rH^\omega$ of the $2n$-dimensional subspace L^{2n} of $L_2(\mathbb{T})$, then the function $\frac{H_{\tau,r}(t)}{|f_r(t)|}$ may be taken over a subset of some subspace of $L_2(\mathbb{T})$ with dimension $\le 8n$ and hence by Lemma 3, we may obtain the following estimates

$$\sup_{\tau\in\mathbb{R}}\int_0^{2\pi}|x_\tau(t)|^2dt \ge d_{8n}^2(k(\varepsilon_r\varphi_0), L_2(\mathbb{T})) = \pi\sum_{k=4n+1}^{+\infty}\frac{1}{(2k+1)^2} > \frac{1}{3n\pi}.$$

Hence, we also obtain

$$\frac{1}{3n\pi} < d_{8n}^2\left(k(\varepsilon_r\varphi_0), L_2(\mathbb{T})\right) \le 2d_{8n}\left(k(\varepsilon_r\varphi_0), \quad L_1(\mathbb{T})\right) \le 2\sup_{\tau\in\mathbb{R}}\int_0^{2\pi}|x_\tau(t)|dt.$$

Thus, by the 2π-periodicity of the function x_τ on the variate τ, there exists a $\tau_0\in[0,2\pi]$ such that

$$\int_0^{2\pi}|x_{\tau_0}(t)|dt \ge \frac{1}{6n\pi}.$$

In Lemma 4, take $A = 2$, $B = \frac{1}{6n\pi}$, $\delta_n = \frac{1}{48n\pi}$, $x = x_{\tau_0}$, $y = |f_r|$ and

$$\Delta = [\gamma_r, \gamma_r+2\pi], \quad D(A,B) = [\gamma_r,\gamma_r+\delta_n]\cup[\pi+\gamma_r-\delta_n, \quad \pi+\gamma_r-\delta_n]\cup[2\pi+\gamma_r-\delta_n, 2\pi+\gamma_r],$$

by the properties of f_r, we have

$$\int_{D(A,B)} dt = 4\delta_n = \frac{B}{A}, \quad \int_0^{2\pi}|x(t)y(t)|dt \ge 2\int_{D(A,B)}|y(t)|dt = 8\int_{\gamma_r}^{\gamma_r+\delta_n}|f_r(t)|dt. \tag{3.14}$$

To give the estimate of $\int_{\gamma_r}^{\gamma_r+\delta_n} |f_r(t)|dt$, we need some properties of f_r. Since the function $|f_r|$ is concave and increasing on the interval $[\gamma_r, \gamma_r + \pi/2]$, which γ_r is a zero of $|f_r|$ and $|f_r\,(\gamma_r + \pi/2)|$ is the maximum value, then there exists an absolutely constant $c > 0$ such that

$$|f_r(t+\gamma_r)| \geq ct, \quad t \in \left[0, \frac{\pi}{2}\right].$$

Thus, we have

$$\int_{\gamma_r}^{\gamma_r+\delta_n} |f_r(t)|dt = \int_0^{\delta_n} |f_r(t+\gamma_r)|dt \geq \int_0^{\delta_n} ctdt = \frac{c\delta_n^2}{2} \gg \frac{1}{n^2}.$$

Further, by (3.14), we conclude that

$$\int_0^{2\pi} |x(t)y(t)|dt \geq 2 \int_{D(A,B)} |y(t)|dt \gg \frac{1}{n^2},$$

which is (3.13). This shows that (3.5) is valid. We complete the proof of Theorem 4.

In the proof of Theorem 5, we need to use the following lemma.

Lemma 5 [1, Lemma 2 in Sect. 84]. *Let f be a continuous function with the period 2π and if there exists a function $\psi \in E_\sigma$ such that $\sup_{x\in\mathbb{R}} |f(x) - \psi(x)| \leq \delta$. Then there is a trigonometric polynomial sum of the form $\phi(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ with $n < \sigma$, for which the relation*

$$\sup_{x\in\mathbb{R}} |f(x) - \phi(x)| \leq \delta.$$

is likewise fulfilled.

Remark of Lemma 5. In the proof of Lemma 5, the sequence $\{\psi_N\}$ of functions defined by

$$\psi_N(x) = \frac{1}{2N+1} \sum_{k=-N}^{N} \psi(x + 2k\pi)$$

was applied. To discuss our problems, here we shortly listed the proof in Achieser's monograph [1, Sect. 84] as follows. By using the facts that in a subset $\mathfrak{M}$ of E_σ if all the functions f in $\mathfrak{M}$ are uniformly bounded on the real axis $\mathbb{R}$, then the functions in $\mathfrak{M}$ are equi-continuous in every bounded point set of complex plane, and hence every sequence in $\mathfrak{M}$ contains a locally uniformly convergent subsequence, we knew that some subsequence $\{\psi_{N_m}\}$ of the sequence $\{\psi_N\}$ is locally uniformly convergent.

Here the so-called a sequence of functions to be "locally uniformly convergent" means that the sequence is uniformly convergent in every bounded point set of complex plane. And the limit function ϕ of $\{\psi_{N_m}\}$ is likewise contained in E_σ and obviously has the period 2π.

The above facts may be seen also from Nikol'skii's monograph [19, Theorem 3.3.6].

Further, for a function $\psi \in E_\sigma \cap W^r H^\omega$, by the above-mentioned process and the well-known Bernstein inequality on the functions E_σ, we may see that $\{\psi_{N_m}^{(r)}\}$ locally uniformly converge to $\phi^{(r)}$ and $\psi \in \tilde{W}^r H^\omega$.

Proof of Theorem 5. Upper estimate. For $\sigma > 0$, let J_σ be the Jackson kernel defined by

$$J_\sigma(x) = \lambda_\sigma \left(\frac{\sin \frac{\sigma x}{4}}{x} \right)^4,$$

where λ_σ is an absolutely constant dependent only on σ. Thus, if $f \in W^r H^\omega$, then it is easy to verify that the convolution $J_\sigma * f$ of J_σ and f is an element of $W^r H^\omega$.

Similar to the period case, we can obtain that for $f \in W^r H^\omega$ there is the following estimate

$$\sup_{x\in\mathbb{R}} |f(x) - J_\sigma * f(x)| \ll \begin{cases} \sigma^{-r}\omega\Big(\dfrac{1}{\sigma}\Big), & r = 0, 1, \\ \sigma^{-2}, \quad r \geq 2, & r \in \mathbb{N}. \end{cases}$$

Lower estimate. By Lemma 5, Remark of Lemma 5, we have the following estimate

$$E(W^r H^\omega, E_\sigma \cap W^r H^\omega)_{L_\infty} \geq E(\tilde{W}^r H^\omega, E_\sigma \cap W^r H^\omega)_{L_\infty}$$

$$= \sup_{f\in\tilde{W}^r H^\omega} \inf \Big\{ \sup_{x\in\mathbb{R}} |f(x) - \psi(x)| : \psi \in E_\sigma \cap W^r H^\omega \Big\}$$

$$= \sup_{f\in\tilde{W}^r H^\omega} \inf \Big\{ \sup_{x\in\mathbb{R}} |f(x) - \psi(x)| : \psi \in E_\sigma \cap \tilde{W}^r H^\omega \Big\}$$

$$= (\tilde{W}^r H^\omega, T_{n+1} \cap \tilde{W}^r H^\omega)_{\tilde{L}_\infty},$$

where $n \in \mathbb{N}, \sigma - 1 \leq n \leq \sigma$.

Sum up, by above discussion and Theorem 4, we complete the proof of Theorem 5.

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